Wed, Sept 10, 2014

Last time: We defined T_xM to be set of derivations of $C^\infty(M) \to \mathbb{R}$.

$\circlearrowleft$ ev_x

ring map
$$ev_x : C^\infty(M) \longrightarrow \mathbb{R}$$
$$f \longmapsto f(x)$$
induces module action of $C^\infty(M)$ on $\mathbb{R}$.

We proved:

- If $f|_U \equiv g|_U$ on an open U s.t. $x \in U$, then $v(f) = v(g)$ $\forall$ $v \in T_xM$
- $T_0\mathbb{R}^n \cong \mathbb{R}^n$.
$$v \longmapsto \sum v(x_i) \frac{\partial}{\partial x_i}\Big|_0$$

<u>Exer</u> Show from the <u>definition</u> that T_xM is a vector space$/\mathbb{R}$.

Using $\mathrm{Der}_k(R,M) \subset \hom_k(R,M)$

<u>Sol'n</u> Let $t \in \mathbb{R}$, $v, v' \in T_xM$. Set $(tv+v')(f) := tv(f) + v'(f)$.

Then $(tv+v')(fg) = tv(f)g + tf\cdot v(g) + v'(f)g + fv'(g) = (tv+v')(f)\cdot g + f(tv+v')(g)$ //

The fact

$$f|_U \equiv g|_U \implies v(f) = v(g)$$

shows a derivation @ x is only sensitive to the local behavior of a function at x.

Consider the equivalence relation

$$f \sim g \iff \exists\, U \subset M \text{ open}, \ x \in U \text{ s.t. } f|_U = g|_U.$$

We call an equivalence class $[f]$ a <u>germ</u> of $\underbrace{\text{a fxn}}_{\text{(smooth)}}$ @ x.

Let $\mathrm{Germ}(x) := C^\infty(M)/_\sim$.

This is still an $\mathbb{R}$-algebra, and

$$\begin{aligned} ev_x : \mathrm{Germ}(x) &\longrightarrow \mathbb{R} \\ [f] &\longmapsto f(x) \end{aligned}$$

is a ring map.

<u>Prop</u>

$$\begin{aligned} T_x M &\longrightarrow \mathrm{Der}\big(\mathrm{Germ}(x), \overset{\circlearrowleft\, ev_x}{\mathbb{R}}\big) \\ v &\longmapsto \big([f] \mapsto v(f)\big) \end{aligned}$$

is an isomorphism of $\mathbb{R}$ vector spaces.

Propn: Let $f : M \to N$ be smooth. $\forall x \in M$, f induces a map of vector spaces

$$T_x f : T_x M \longrightarrow T_{f(x)} N$$

Moreover,

$$T_x(g \circ f) = T_{f(x)} g \circ T_x f.$$

Pf

Define

$$T_x f(v)(h) = v(h \circ f).$$

i.e.,

$$\begin{array}{ccccc} C^\infty(N) & \xrightarrow{f^*} & C^\infty(M) & \xrightarrow{v} & \mathbb{R} \\ h & \longmapsto & h \circ f & \longmapsto & v(h \circ f) \end{array}$$

Since $h \mapsto h \circ f$ is a ring map, $T_x f(v)$ is a derivation:

$$\begin{aligned} v(h_1 h_2 \circ f) &= v(h_1 \circ f \cdot h_2 \circ f) \\ &= v(h_1 \circ f) \cdot h_2 \circ f + h_1 \circ f \cdot v(h_2 \circ f) \\ &= T_x f(v)(h_1) \cdot \underbrace{h_2(f(x))}_{ev_{f(x)}} + h_1(f(x)) \cdot T_x f(v)(h_2) \end{aligned}$$

Finally, $(T_x(g \circ f)(v))(h) = v(h \circ g \circ f)$, and

Chain rule is obvious using algebraic definition

$$(T_{f(x)} g \circ T_x f)(v)(h) = T_x f(v)(h \circ g) = v(h \circ g \circ f). \quad //$$

Ex $\forall\ x \in \mathbb{R}^n$,

translation by x and $-x$

gives an isomorphism

$$\begin{tikzcd} T_0\mathbb{R}^n \arrow[r, "+x", "\simeq"'] \arrow[rr, bend right, "\mathrm{id}"'] & T_x\mathbb{R}^n \arrow[r, "-x", "\simeq"'] & T_0\mathbb{R}^n \end{tikzcd}$$

sending

$$\left.\frac{\partial}{\partial x_i}\right|_0 \longmapsto \left.\frac{\partial}{\partial x_i}\right|_x .$$

Next goal: Define a vector bundle, TM, called the <u>tangent bundle</u>.

Prove:

<u>Thm</u>

$$\mathrm{Der}_{\mathbb{R}}(C^\infty(M), C^\infty(M)) \cong \Gamma(TM).$$

The tangent bundle:

Fix $U \subset \mathbb{R}^n$ open. We identify

$$\bigcup_{x \in U} \{x\} \times T_x U \cong U \times \mathbb{R}^n$$

by

$$(x, v) \longmapsto (x, \vec{a})$$

where

$$v = \sum a_i \frac{\partial}{\partial x_i}\Big|_x .$$

If M is a smooth manifold, (U, ϕ) a chart, we have a map

$$\tilde{U} := \bigcup_{x \in U} \{x\} \times T_x M \longrightarrow \phi(U) \times \mathbb{R}^n$$

$$(x, v) \longmapsto \big(\phi(x),\ v(x_i \circ \phi)\big)$$

Ex: $M = \mathbb{R}^n$, $(U, \phi) = (\mathbb{R}^n, id)$. Then $T_0\mathbb{R}^n \to \mathbb{R}^n$, $v \mapsto \vec{a}$, $v = \sum a_i \frac{\partial}{\partial x_i}\Big|_0$.

Call this map $\tilde{\Phi}$.

Then

$$\tilde{\Phi}_\beta \circ \tilde{\Phi}_\alpha^{-1}(x, \vec{a})$$

$$= \left(\phi_\beta \circ \phi_\alpha^{-1}(x), D(\phi_\beta \circ \phi_\alpha^{-1})\Big|_x (\vec{a}) \right)$$

Pf Let $v_i \in T_x M$ such that

$$\tilde{\Phi}_\alpha(x, v_i) = \left(\phi_\alpha(x), \frac{\partial}{\partial x_i}\right) \in \bigcup_{x \in U} \{\phi_\alpha(x)\} \times T_{\phi_\alpha(x)} \mathbb{R}^n .$$

Then $\tilde{\Phi}_\beta \circ \tilde{\Phi}_\alpha^{-1}\left((\phi_\alpha(x), \frac{\partial}{\partial x_i})\right) = \tilde{\Phi}_\beta\left((x, v_i)\right)$

$$= \left(\phi_\beta(x), \vec{b}\right)$$

where $b_j = v_i(x_j \circ \phi_\beta)$

$$= T_{\phi_\alpha(x)} \phi_\alpha^{-1}\left(\frac{\partial}{\partial x_i}\right)(x_j \circ \phi_\beta)$$

$$= \frac{\partial}{\partial x_i}\left(x_j \circ \phi_\beta \circ \phi_\alpha^{-1}\right)$$

$$= D(\phi_\beta \circ \phi_\alpha^{-1})_{ij} \ //$$

Since $\phi_\beta \circ \phi_\alpha^{-1}$ are all C^∞ by defn of smooth atlas, the fnction

$$\phi_\beta \circ \phi_\alpha^{-1} \times \underbrace{D(\phi_\beta \circ \phi_\alpha^{-1})}_{n \times n \text{ matrix w/ } C^\infty \text{ entries}}$$

is smooth.

Cor $\{(\tilde{U}_\alpha, \tilde{\Phi}_\alpha)\}_\alpha$ is a smooth atlas for $TM := \bigcup_{x \in M} T_x M \times \{x\}$

To topologize:

$$TM := \coprod_{\alpha} \phi_{\alpha}(U_{\alpha}) \times \mathbb{R}^n \Big/ \sim_{\text{charts}}$$

Can check $\tilde{U}_{\alpha} \subset TM$ are open.

__Prop__ The map

$$\begin{aligned} TM &\xrightarrow{\pi} M \\ (x, v) &\longmapsto x \end{aligned}$$

is smooth.

__Pf__ First check it's continuous.

Well,

$$\pi^{-1}(U) = \tilde{U} \leftarrow \text{open}$$

$\forall$ small open balls $U \subset M$.

To check smoothness:

$$\psi \circ \pi \circ \tilde{\phi}^{-1}$$

sends

$$(x, a) \mapsto \left(\phi^{-1}(x), T_x\phi^{-1}(a) \right)$$

$$\xmapsto{\pi} \phi^{-1}(x)$$

$$\xmapsto{\psi} \psi \circ \phi^{-1}(x)$$

is C^{∞} by defn of smooth atlas

on M. //