Mon. Sept 8, 2014

See online addendum re: paracompact $\iff$ 2^{nd} Countable.

Today: tangent spaces.

<u>Defn</u> $f : X \to Y$ continuous.

f is <u>smooth</u> if

$\psi \circ f \circ \phi^{-1}$ is smooth

$\forall$ charts $(U, \phi) \in A_X$
$(V, \psi) \in A_Y$

(wherever the composition is defined).

w/o assumption that f be continuous, definition is terrible.

<u>Ex</u> (John Lee) Fix $U_0 \subset X$

$$f : X \longrightarrow (\mathbb{R}, A)$$

$$x \longmapsto \begin{cases} 1 & x \in U_0 \\ 0 & \text{otherwise.} \end{cases}$$

Obviously not continuous. But cover $\mathbb{R}$ by opens V s.t. V doesn't contain both 0 and 1.

Then $\psi \circ f \circ \phi^{-1}$, defined on

$$\phi(f^{-1}(V))$$

is always a constant map, hence is smooth!

<u>Rmk</u> Composition of smooth maps is smooth.

<u>Rmk</u> $\forall$ charts (U, ϕ), the maps

$$\phi(U) \to U \to X$$
$$\searrow \phi(U)$$

are all smooth maps.

Defn Let R be a k-algebra.
(R, k are both unital and commutative.)
Fix an R-module M.
A k-derivation of M is a k-linear map
$$D: R \longrightarrow M$$
such that $D(f \cdot g) = \underbrace{Df \cdot g}_{\substack{g \in R \\ \text{acting} \\ \text{on } Df \in M}} + f \cdot Dg$.

i.e., D satisfies Leibniz rule.

Ex Let X be a vector field on $\mathbb{R}^n$. Let $k = \mathbb{R} \cong$ constant fxns on $\mathbb{R}^n$
$R = C^\infty(\mathbb{R}^n) =$ smth fxns on $\mathbb{R}^n$
$M = R$.

Then let $D = D_X$ = directional derivative.
The map
$$R \longrightarrow M$$
$$f \longmapsto D_X(f)$$
is a derivation, since
$$D_X(tf + g) = tDf + Dg$$
$$D_X(f \cdot g) = D_X f \cdot g + f \cdot D_X g.$$

Fix $x \in M$

Let

smooth, $\mathbb{R}$-valued fxns

$$ev_x : C^\infty(M) \longrightarrow \mathbb{R}$$
$$f \longmapsto f(x)$$

be evaluation at x.

This is a ring map, so makes $\mathbb{R}$ into a module over $C^\infty(M)$.

__Defn__ A __tangent vector__ to M at x is an $\mathbb{R}$-derivation

$$V : C^\infty(M) \longrightarrow \mathbb{R} \circlearrowleft^{ev_x} C^\infty(M).$$

A smooth vector field on is an $\mathbb{R}$-derivation

$$X : C^\infty(M) \longrightarrow C^\infty(M).$$

__Exer__ Using ev_x, every vector field on M determines a tangent vector at x.

__Pf__ $C^\infty(M) \xrightarrow{X} C^\infty(M) \xrightarrow{ev_x} \mathbb{R}$

$$(ev_x \circ X)(fg) = ev_x(Xf \cdot g + f \cdot Xg)$$

since ev_x is a ring map

$$= ev_x(Xf) \cdot ev_x(g) + ev_x(f) \cdot ev_x(Xg)$$
$$= ev_x \circ X(f) \cdot g + f \cdot ev_x \circ X(g)$$

(Defn of module action induced by ring map.)

Rmk $\mathrm{Der}_k(R, M) \subset \mathrm{Hom}_k(R, M)$
is obviously a k
vector space.

Defn
$$T_x M := \mathrm{Der}_{\mathbb{R}}\left(C^\infty(M), \underset{\circlearrowleft}{\overset{ev_x}{}}\mathbb{R}\right).$$

algebraic characterization (using derivations) of a geometric idea. (tangents to $0 \in \mathbb{R}^n$ are $\cong$ to $\mathbb{R}^n$.)

Thm
$T_0\mathbb{R}^n$ is an n-dim. vector space$/\mathbb{R}$.

fails for C^r manifolds w/ $r<\infty$. $\mathrm{Der}_{\mathbb{R}}(C^r(\mathbb{R}^n), \mathbb{R})$ isn't even finite dimensional.

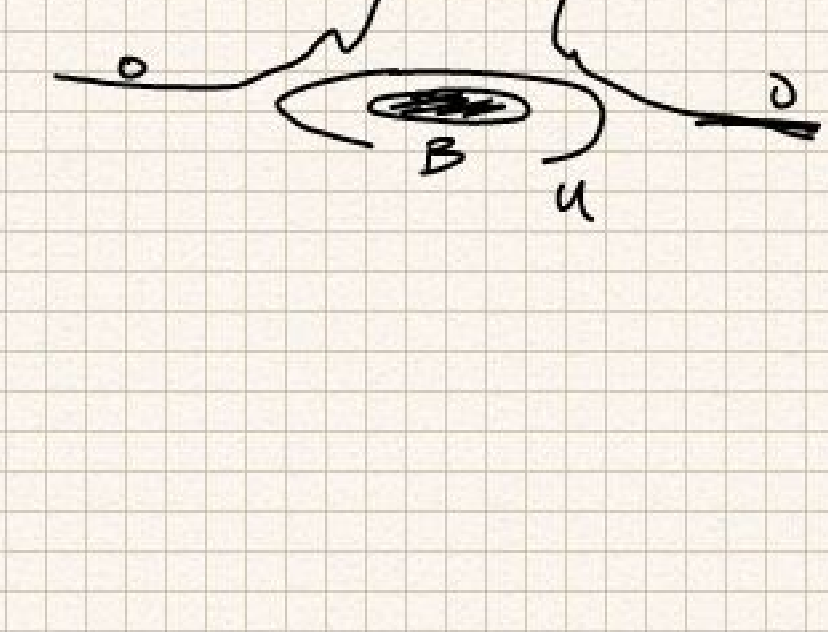
To prove this, we'll use:

Lemma Let $B \subset \mathbb{R}^n$ be a closed ball, $B \subset U$ open.
$\exists$ smooth $h : \mathbb{R}^n \to \mathbb{R}$
s.t.

(1) $h|_B \equiv 1$

(2) $h|_{U^c} \equiv 0$

(3) $h(\mathbb{R}^n) \subset [0,1]$.



Cor Let M be smth mfld, $U \subset M$ open, $f : U \to \mathbb{R}$ smooth.
$\forall$ $K \subset U$ compact, $\exists$
$\tilde{f} : M \to \mathbb{R}$ smooth
s.t. $\tilde{f}|_K = f|_K$.

Pf Cover U by local charts, then cover K by closed, finite-radius balls B_i in these charts. Choose bump functions h_i so $\exists$

$B_i \subset V_i \subset \overline{V_i} \subset U$,

$h_i|_{V_i^c} \equiv 0$.

Pass to finite subcovers, so we have closed balls

$$\phi_i^{-1}(B_i) \subset M$$

w/ fxns $\tilde{h}_i : M \longrightarrow \mathbb{R}$

where $$\tilde{h}_i(x) = \begin{cases} h_i \circ \phi_i(x) & x \in \phi_i^{-1}(\mathbb{R}) \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$h = 1 - \prod(1 - \tilde{h}_i).$$

If $\tilde{h}_i(x) = 1$,

$h(x) = 1. \Rightarrow h|_K \equiv 1.$

If $\tilde{h}_i(x) = 0 \ \forall i$,

$h(x) = 0$

In particular, $h|_{U^c} \equiv 0$

Then

$$\tilde{f}(x) = \begin{cases} 0 & x \notin U \\ f(x)h(x) & x \in U. \end{cases}$$

//

Propn Let M be smth mfld, $x \in M$.

(1) $\forall\ v \in T_xM$,

$v(\text{cnstnt fxn}) = 0$.

(2) If $f, g \in C^\infty(M)$ and

$f|_U = g|_U$

for some neighborhood U of x,

$v(f) = v(g)\ \forall\ v \in T_xM$.

Pf. (1) Let $1 \in C^\infty(M)$ be the unit.

Then $v(1 \cdot 1) = v(1) \cdot 1 + 1 \cdot v(1)$

$= 2v(1)$

Hence $v(1) = 2v(1) \Rightarrow v(1) = 0$.

(2) Let $V \subset U$ be open st $x \in V$ and $\overline{V} \subset U$ is compact. Fix bump fnction $h : M \to \mathbb{R}$ st $h|_{\overline{V}} \equiv 1$, $h|_{U^c} \equiv 0$. Then $h(f-g) \equiv 0$.

Then

$$v(h(f-g)) = v(h) \cdot (f-g) + h \cdot v(f-g)$$

$$= v(h) \cdot (f-g)(x) + h(x) \cdot v(f-g) \quad \text{defn of } ev_x \text{ module action}$$

$$= v(h) \cdot 0 + 1 \cdot v(f-g) \quad f \equiv g \text{ on } U \ni x,\ h|_{\overline{V}} \equiv 1,\ x \in \overline{V}.$$

$$= v(f) - v(g) \quad \text{Linearity of } v$$

$\Rightarrow v(f) = v(g)$. //

Pf of thm

On $\mathbb{R}^n$, we have n derivations @ 0:

$$\frac{\partial}{\partial x_i} : f \mapsto \frac{\partial f}{\partial x_i}(0).$$

(Product, or Leibniz rule, proves derivation property.)

These are linearly independent because

$$\frac{\partial}{\partial x_i} : x_j \mapsto \delta_{ij}.$$

Claim: They span $T_0\mathbb{R}^n$?

NTS $\exists$ a_i, $i=1,\dots,n$ st

$$v(f) = \sum a_i \frac{\partial f}{\partial x_i}(0) \quad \forall f.$$

By Taylor's theorem,

$$f = f(0) + \sum x_i h_i \quad \text{for some } C^\infty \text{ fxn } h_i,\ h_i(0) = \frac{\partial f}{\partial x_i}(0).$$

constant fxn

So

$$v(f) = \sum_{i=1}^{n} v(x_i)\cdot h_i(0) + x_i(0)\cdot v(h_i)$$

$$= \sum v(x_i)\cdot h_i(0)$$

$$= \sum v(x_i)\, \frac{\partial f}{\partial x_i}(0)$$

$$= \left(\sum v(x_i)\, \frac{\partial}{\partial x_i}\right) f.$$

$$\Longrightarrow \quad v = \sum a_i \frac{\partial}{\partial x_i} \quad \text{where } a_i = v(x_i) \quad //$$

Taylor's theorem:

$$\gamma_x : \mathbb{R} \longrightarrow \mathbb{R}^n$$
$$t \longmapsto tx$$

$$f(x) = f(0) + \int_0^1 \frac{\partial}{\partial t}(f\circ\gamma_x)\, dt$$

$$= f(0) + \int_0^1 \sum \frac{\partial f}{\partial x_i}(\gamma_x(t)) \cdot \frac{\partial (x_i \circ \gamma_x)}{\partial t}\, dt$$

$$= f(0) + \sum x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(\gamma_x(t))\, dt}_{h(x)}$$

If $x=0$, $\int_0^1 \frac{\partial f}{\partial x_i}(0)\, dt = \frac{\partial f}{\partial x_i}(0)$.

So requires Taylor's theorem to prove algebraic definition of $T_0\mathbb{R}^n$ is even finite-dimensional!