FRIDAY, SEPT 5, '14

Apologies from last time:
Didn't realize most here have never taken a diff geom class before.
Undergraduate differential geom. is NOT a prereq.

So, a baby example of differential geometry for 2-manifolds.

Basic geometric objects in $\mathbb{R}^2$:

pts $\longleftrightarrow$ elements

lines $\longleftrightarrow$ "shortest" path

circles $\longleftrightarrow$ locus of equidistant pts

triangles $\longleftrightarrow$ piecewise geodesic curve w/ three vertices

all require notion of distance.

Some facts:

For a triangle $T \subset \mathbb{R}^2$,

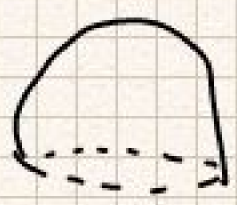
- $\sum$ interior angles $= \pi$
- T similar to T' $\not\Rightarrow T \cong T'$
- Using straightedge and compass, can trisect any edge.

Consider

$$V = \{(x,y,z) \text{ s.t. } z > 0 \text{ and } x^2+y^2+z^2 = 1\}$$

$$\subset S^2$$

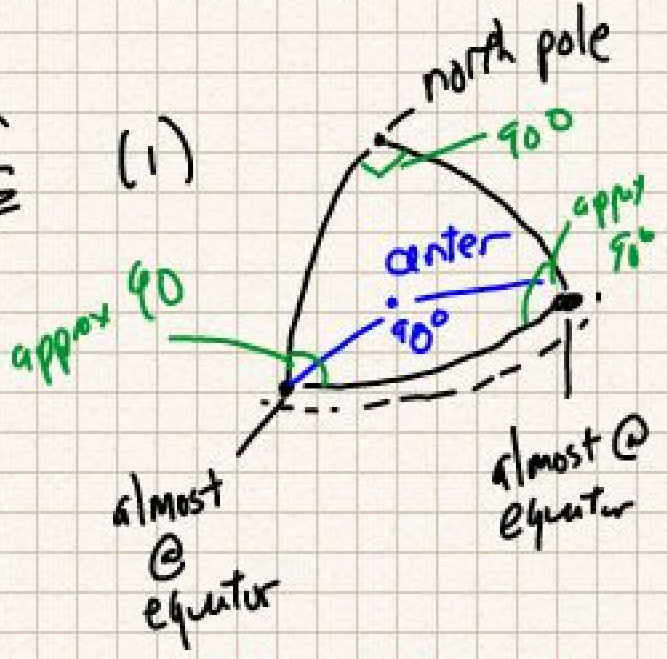
$$\subset \mathbb{R}^3$$



Given $p, q \in V$, $\exists!$ semicircle passing through them. Can prove semicircles are shortest path — i.e., geodesics.

Prop'n None of the facts about triangles in $\mathbb{R}^3$ from before are true on V.

Pf (1)



(2) omitted.

(Napier's laws, analogues of laws of (co)sine)

(3) Galois thy.

Trisect line on V

$\Leftrightarrow$ Trisect angle in $\mathbb{R}^3$

Okay, so there are many geometries out there.

Note $V \cong \mathbb{R}^2$ as a topological space, even as a smooth manifold.

While V inherited geometry from $\mathbb{R}^3$, (extrinsically) we'll see how to put structures on $\mathbb{R}^2 \cong V$ w/o reference to any embedding (intrinsically)

Back to scheduled programming.

Last time:

Defn A smooth n-mfld is a pair

$$(X, \mathcal{A})$$

where

- X is a Hausdorff paracompact space

and

- $\mathcal{A}$ is a collection of charts

$$\{(U_\alpha, \phi_\alpha)\}_\alpha$$

open in X — U_α; $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ homeo onto image

such that

- $X = \bigcup U_\alpha$, and
- $\phi_\beta \circ \phi_\alpha^{-1} \Big|_{\phi_\alpha(U_\alpha \cap U_\beta)}$ is a smooth fxn.

Defn A space X is <u>second countable</u> if $\exists$ a countable base for the topology of X.

(ie, a collection $\mathcal{B}$ of opens s.t. $\forall$ $U \subset X$ open, $\forall x \in U$, $\exists$ $V \in \mathcal{B}$ s.t. $V \subset U$.)

Ex $\mathbb{R}^n$. Take

$$\mathcal{B} = \{ B(x,r) \text{ s.t. } x_i \in \mathbb{Q},\ r \in \mathbb{Q} \}.$$

ie, balls of rational radius centered at pts w/ rational coords

Ex Any subspace of a second countable space.

If $Y \subset X$, let

$$\mathcal{B}_Y = \{ U \cap X \mid U \in \mathcal{B}_X \}$$

Defn A space X is called <u>paracompact</u> if any open cover admits a locally finite refinement.

Recall: An open cover $\{V_\beta\}$ is a <u>refinement</u> of $\{U_\alpha\}$ if $\forall \beta, \exists \alpha$ s.t.

$$V_\beta \subset U_\alpha.$$

(Smaller opens!)

A cover is <u>locally finite</u> if $\forall x \in X$, ~~$\exists$ only finitely many β s.t. $x \in V_\beta$~~ $\exists \alpha$ s.t. $U_\alpha \cap V_\beta \neq \phi$ for only finitely many β.

Harder to prove $\mathbb{R}^n$ is paracompact.

Thm If X is locally compact and Hausdorff, X is second countable iff it is paracompact.

Proof later.

Why do we want this fact?

Often, to prove a certain structure exist, on a mfld, we'll prove that it exists on $\mathbb{R}^n$ (and open subsets thereof). Then we'll paper-machet, or patch, the structures from $\mathbb{R}^n$ to all of the mfld.

A priori, a point x may have infinitly many charts, each with a specified structure, and we'll need to reconcile them in a compatible way. Rather than pigeonholing an infinite family of structures, we can play w/ a <u>finite</u> collection using paracompactness.

Some examples:

__Ex__ $\mathbb{R}^n$ with

$$\mathcal{A} = \{(\mathbb{R}^n, \mathrm{id})\}.$$

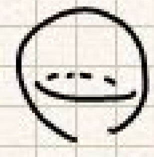
__Ex__ $S^n \subset \mathbb{R}^{n+1}$.

Stereographic projection from n pole + s pole.

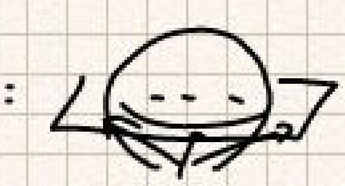
$x_+ = (1, 0, \dots, 0)$

$x_- = (-1, 0, \dots, 0)$.

$U = S^n \setminus \{x_+\} =$ 

$V = S^n \setminus \{x_-\} =$ 

ϕ :  $: U \to \mathbb{R}^n$

ψ :  $: V \to \mathbb{R}^n$

Check explicitly that

$$\phi \circ \psi^{-1} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$$
$$\psi \circ \phi^{-1} : \quad " \quad \longrightarrow \quad "$$

are smooth.

__Ex__ Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be smooth. Suppose

$$df_x : T_x\mathbb{R}^n \longrightarrow T_{f(x)}\mathbb{R}^m$$

is a surjection $\forall x$. Then $\forall y \in \mathbb{R}^m$, $f^{-1}(y)$ is a smooth mfld.

overkill!

(Charts given by inverse fxn thm.)

(Exercise in next p-set.)

Defn Let

$$(X, \mathcal{A}_X)$$

and

$$(Y, \mathcal{A}_Y)$$

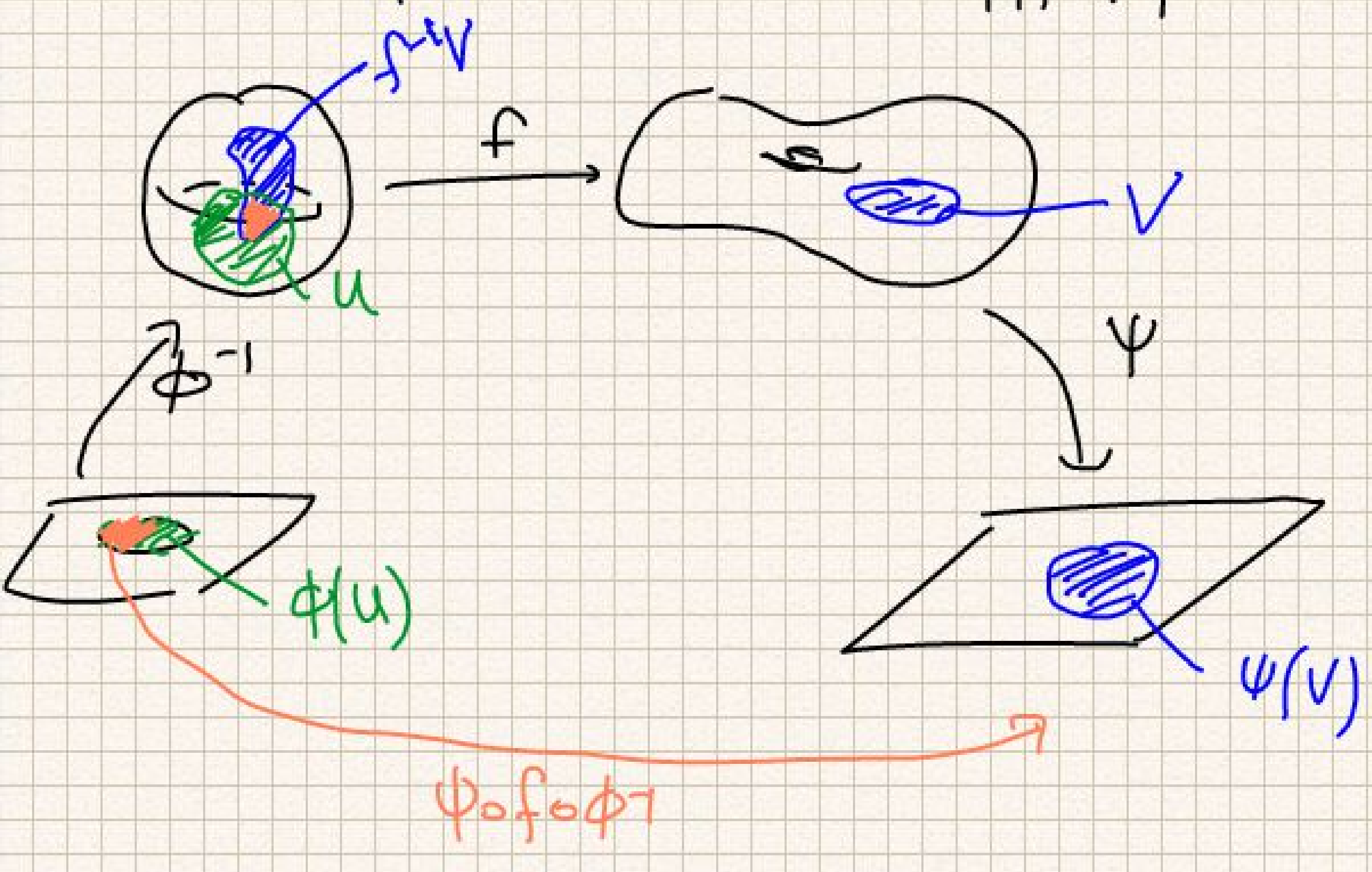
be smooth mflds, possibly of diffent dimensions.

A ~~fuction~~ contiuous fuction

$$f : X \to Y$$

is called <u>smooth</u> if

$$\forall \ (U, \phi) \in \mathcal{A}_X, \ (V, \psi) \in \mathcal{A}_Y, \quad \psi \circ f \circ \phi^{-1}\Big|_{\phi(f^{-1}V)} \text{ is smooth.}$$



~~Rmk Any smooth f turns out to be continuous. (Easy exercise.)~~

If you demand that $\forall \ p \in X$, $\exists \ (U, \phi), (V, \psi)$ s.t. $f(U) \subset V$, $p \in U$, and $\psi \circ f \circ \phi^{-1}$ is smooth, f is C^0 and smooth.

__Rmk__ In homework, you'll address the question of when two atlases on X should be considered equivalent.

__Defn__ A homeomorphism

$$f : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$$

is called a diffeomorphism if both f, f^{-1} are smooth.

__Thm__ (Donaldson, Freedman, Kirby)

$\exists$ uncountably many diffeomorphism types of $\mathbb{R}^4$.

__Thm__ (Milnor, Kervaire-Milnor)

$\exists$ more than one (but finitely many) diffeomorphism type for S^7, and for many other spheres.

Gotta get back to material!

What IS a tangent vector at a point $x \in X$?

There are some nice interpretations:

(1) If $X \subset \mathbb{R}^n$, the obvious one. (Vector in $\mathbb{R}^n$ based at x, tangent to X.)

(2) More intrinsic, still geometric:
"The derivative" of a curve $\gamma: \mathbb{R} \to X$ at a point.

(3) Intrinsic, purely algebraic:
A way of taking a derivative.

We'll take (3) as our approach. It's beautiful, and applies to algebraic geometry, too.

Defn Let C be a R-algebra. Fix a map of R-algebras

$$C \xrightarrow{e} R.$$

A *derivation at e* is a map of R-modules

$$D : C \longrightarrow R$$

such that

$$D(fg) = Df \cdot g + f \cdot Dg$$

e makes R into a C-module, so $\cdot$ is the action

Ex $C = C^\infty(\mathbb{R}^n)$ (= smth fns on $\mathbb{R}^n$)

$R = \mathbb{R}$.

$\mathbb{R} \curvearrowright C^\infty$ by scaling.

Fix $x \in \mathbb{R}^n$. Define

$$ev_x : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$$
$$f \longmapsto f(x).$$

Given any vector $\vec{v}$ at x, the directional derivative

$$D_{\vec{v}} : C^\infty(\mathbb{R}^n) \to \mathbb{R}$$

is a derivation.

Thm $\exists$ isomorphism of vector spaces

$$\text{Derivations at } ev_0 \cong \text{Tangent vectors to } 0 \text{ in } \mathbb{R}^n.$$

Pf:

We have map

$$\vec{v} \longmapsto D_{\vec{v}}.$$

$\{\vec{v}\}$ has basis

$$\left\{ \frac{\partial}{\partial x_i} = \begin{matrix}\text{tan. vector} \\ \text{in } x_i \\ \text{direction}\end{matrix} \right\}.$$

These remain linearly independent, since the coordinate functions

$$x_i : \mathbb{R}^n \to \mathbb{R}$$

satisfy

$$D_{v_i} x_j = \delta_{ij}.$$

Image spans by Taylor's theorem. //