Differential Geometry II

What is the object of study?

Differential Geometry

must involve calculus somehow!

Smooth manifolds (as opposed to topological mflds. More soon.)

Meaning has expanded over time.

Obvious answer:

metry $\longleftrightarrow$ metrics of some sort.

$\langle x,x\rangle \geq 0$ ← Riemannian geometry.

$\begin{pmatrix} -1 & 0 \\ 0 & \ddots & 1 \end{pmatrix} \longleftrightarrow$ Lorentzian

Also: study of other structures (not just metrics).

Principle bundles
+
Connections

(we'll get to this).

Today: Nothing too formal. Heuristic examples of mflds, representative theorems of differential geometry.

Let's start w/ a representative example from more classical diff geom.

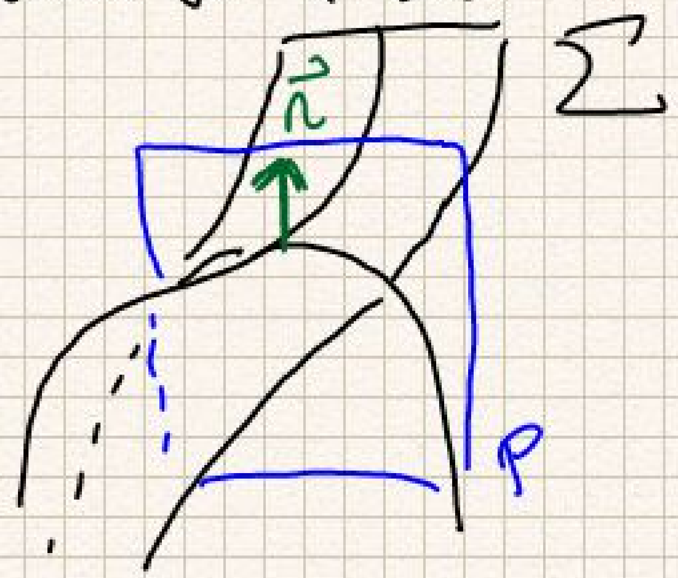
<u>Thm</u> (Baby Gauss-Bonnet)

Let Σ be a regular, compact, orientable surface in $\mathbb{R}^3$. Then

$$\int_\Sigma K \, dA = 2\pi \underbrace{\chi(\Sigma)}_{\text{Euler characteristic}}.$$

Gaussian curvature

Let's review for a second.



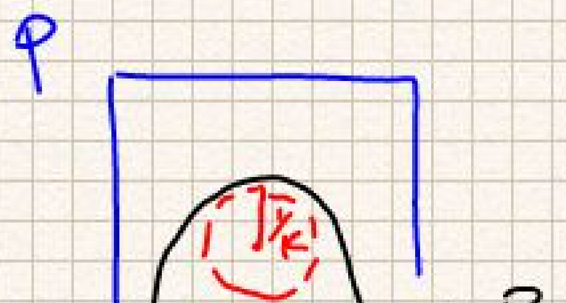
How is K defined?

At a point $x \in \Sigma$, we have a normal vector $\vec{n}$.

Choose any plane P containing x and $x+\vec{n}$.

Then $\Sigma \cap P$ is a curve γ_P, and we can compute its curvature K_{γ_P} at x, as a curve in P.

$\frac{1}{R}$ of circle approximating γ_P.

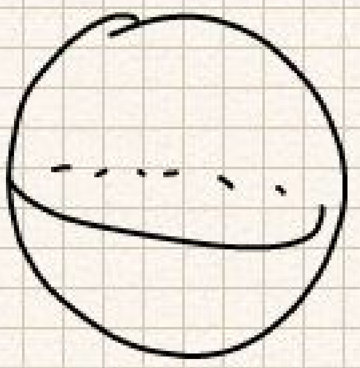


We then have

$$K := \left(\max_P K_{\gamma_P}\right)\left(\min_P K_{\gamma_P}\right)$$

$$\left(= \det\left(2^{nd} \text{ fnd. form}\right)\right).$$

Ex $\Sigma = S^2_R = \partial B^3_R$



Then K_{γ_P} is constant $\forall P$ $\forall x$,

since

$\Sigma \cap P =$ great circle of radius R

and

$$K_{\gamma_P} = \frac{1}{R}.$$

Hence
$$K(x) = \max K_{\gamma_P} \cdot \min K_{\gamma_P}$$
$$= \frac{1}{R} \cdot \frac{1}{R}$$
$$= 1/R^2.$$

$$\int_{S^2_R} K \, dA = \frac{1}{R^2} \cdot \int_{S^2_R} K \, dA$$
$$= \frac{1}{R^2}\left(4\pi R^2\right)$$
$$= 4\pi$$
$$= 2\pi \cdot 2$$
$$= 2\pi \, \chi(S^2).$$

What were the ingredients?

- Inner product on $\mathbb{R}^3$.
  More specifically, at every point $x \in \Sigma$, we made use of the inner product for vectors emanating from x, to define $\vec{n}$.

- Gauss's Theorema Egregium says one can get rid of embedding into $\mathbb{R}^3$, and study what we have left on Σ. We'll no longer have things like $\vec{n}$, but we will have
  - notion of tangent vectors to $x \in \Sigma$, $T_x\Sigma$
  - an inner product on $T_x\Sigma$ $\forall x$.

Σ will be a manifold, rather than an embedded manifold in $\mathbb{R}^3$

Later, we will define the bundle of all the tangent spaces of a mfld, the "tangent bundle" $T\Sigma$.

A structure on $T_x\Sigma$ $\forall x$.

Thm Given an inner product g_x on $T_x\Sigma$ $\forall x$ varying smoothly, one can define a fxn K st

$$\int_\Sigma K \, dA = 2\pi \chi(\Sigma).$$

Moreover, if g_x is induced by an embedding $\Sigma \subset \mathbb{R}^3$, the two definitions of K agree.

Three themes:

- Structure on $T\Sigma$.

  In this case, a Riemannian metric — ie, positive definite inner product on $T_x\Sigma$ $\forall x \in \Sigma$. (Geometry)

  Later, we'll talk about other kinds of structures like connections on principle bundles.

- Intrinsic v Extrinsic

  Σ on its own i.e., Σ as a manifold. | Σ as an embedded subobject of something else ($\mathbb{R}^3$)

  Can still talk about derivatives, and tangent vectors. (Differential)

- Topology. Strictly speaking, topology is the study of geometry w/out the measurements — so of shapes, not of sizes. Yet topology will fundamentally appear throughout, as geometry + topology aide each other.

  (Not in phrase "differential topology")

  For instance, what shapes allow certain structures?

Corollary If Σ is $\overbrace{\text{compact and}}^{\text{orientable}}$ admits a Riemannian metric s.t.

$$
\begin{array}{lll}
K \equiv 0 & & \chi(\Sigma) = 0 \quad (\Rightarrow \Sigma \cong T^2) \\
K(x) > 0 \;\; \forall x & \text{then} & \chi(\Sigma) > 0 \quad (\Rightarrow \Sigma \cong S^2) \\
K(x) < 0 \;\; \forall x & & \chi(\Sigma) < 0 \quad (\Rightarrow \text{genus} \geq 2)
\end{array}
$$

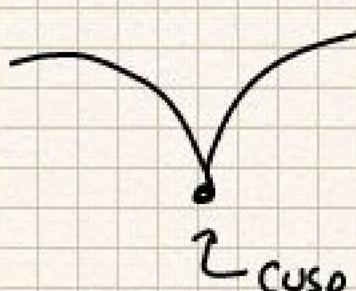
[What is a topological mfld?]

Defn Fix $n \geq 0$.
A topological space X is called a <u>topological manifold of dimension n</u> if it is (adjectives) and locally homeomorphic to $\mathbb{R}^n$.

i.e., $\forall\, x \in X$,
$\exists\, U \subset X$ open, $x \in U$
s.t. $U \cong \mathbb{R}^n$
as a topological space.

Ignore "adjectives" for now.
I'd rather motivate "topological."

Ex



$= X \subset \mathbb{R}^2$

X is a topological 1-mfld.

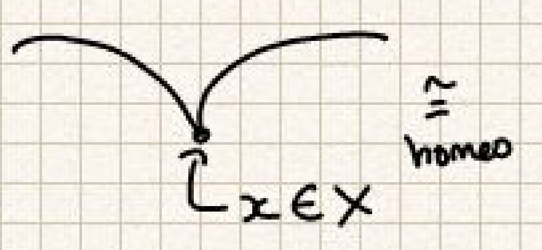
Ex

$= X$

(nothing rigorous yet).

This alone doesn't tell us how to do calculus — ie, take derivatives.

For instance, how do you know when a function $X \to \mathbb{R}$ is smooth for $x \in X$?



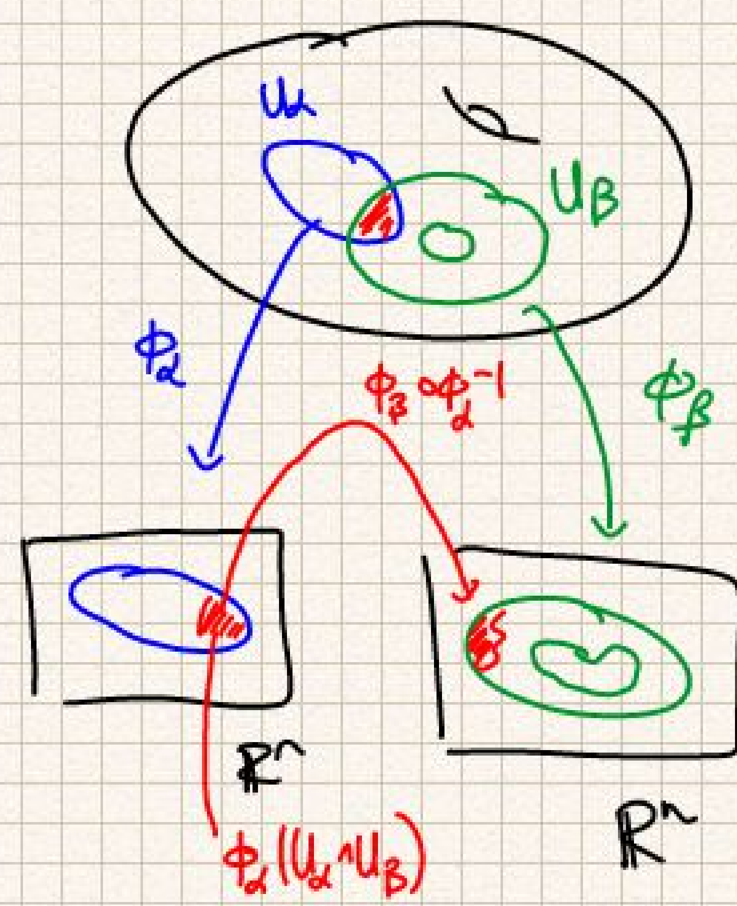
Lesson: not enough to know X as a topological space.

<u>Defn</u> Let X be a topological n-mfld.

- A <u>chart</u> for X is a pair $(U_\alpha, \varphi_\alpha)$ where $U_\alpha \subset X$ open, $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$ is a homeom. onto its image.

- An <u>atlas</u> for X is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ s.t. $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = X$.

  NOT a $\coprod_\alpha$, so U_α can intersect $U_{\alpha'}$.



- An atlas is called C^k if $\forall\, \alpha, \beta$
$$\varphi_\beta \circ \varphi_\alpha^{-1}\Big|_{\varphi_\alpha(U_\alpha \cap U_\beta)}$$
is C^k.

(Since $\varphi_\alpha^{-1} \circ \varphi_\beta$ is a fn from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$, we know what it means for it to be C^k.)

Defn A <u>C^k-differentiable n-manifold</u> is a pair

$$(X, \mathcal{A})$$

where $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ is a C^k atlas for X, and X is a topological n-mfld.

A C^∞-differentiable n-mfld is called a <u>smooth</u> n-mfld.

Ex

- $X = \mathbb{R}^n$

  $\mathcal{A} = \{ (U = \mathbb{R}^n, \phi = id) \}$

  a singleton set.

  Chart condition: $U \cap U = X$,

  $$\mathbb{R}^n \xrightarrow[\underset{id}{\shortparallel}]{\phi \circ \phi^{-1}} \mathbb{R}^n$$

  is smooth.

- Likewise, if $X = U \subset \mathbb{R}^n$,

  set $\mathcal{A} = \{ (U, U \hookrightarrow \mathbb{R}^n) \}$.

- $X = S^n$. Let

  $U_1 = S^n \setminus$ north pole

  $U_2 = S^n \setminus$ south pole

  $\phi_i : U_i \longrightarrow \mathbb{R}^n$ stereographic projection

check for yourself that $\phi_i \circ \phi_j^{-1}$ is C^∞.

Ex Let (X, A) be a smth mfld, and $V \subset X$ open. Define

$$A_V = \{ (U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V}) \}.$$

Then (V, A_V) is a C^∞ mfld.

Rmk We've disregarded "adjective," but these examples satisfy "adjective."

We will often just write

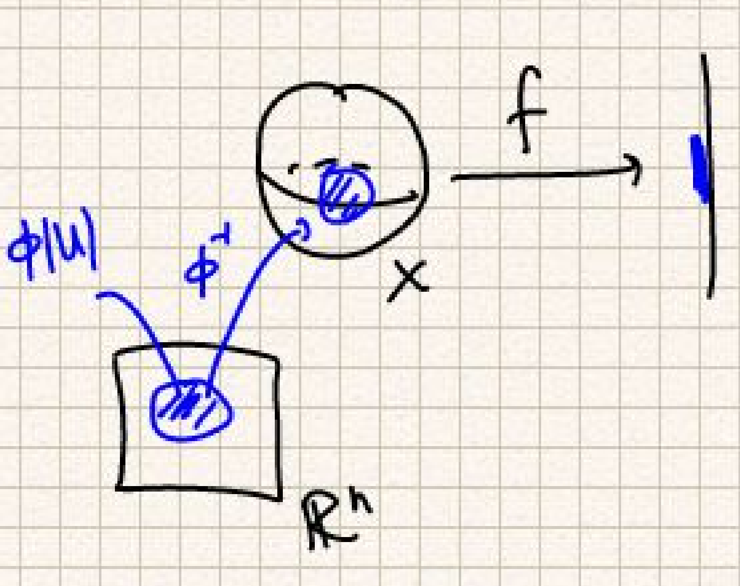
$$X$$

with the atlas A implicit.

Finally!

Defn Let $X = (X, A)$ be a smooth n-mfld. A function $f : X \to \mathbb{R}$ is called smooth if

$$f \circ \phi^{-1} : \phi(U) \longrightarrow \mathbb{R}$$



is smooth $\forall \ (U, \phi) \in A$.

Defn Let $X = (X, A_X)$
$$Y = (Y, A_Y)$$
be two smooth mflds, possibly of differt dimensions. A function
$$f : X \to Y$$
is called <u>smooth</u> if

$$g : Y \to \mathbb{R} \text{ smooth}$$

$$\Rightarrow \quad g \circ f : X \to \mathbb{R} \text{ smooth.}$$

Exer
$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

f, g smooth $\Rightarrow$ $g \circ f$ smooth.

$\forall\ h : Z \to \mathbb{R}$,

$h \circ g$ smth since g is

$(h \circ g) \circ f$ smth since f is
$\parallel$
$h \circ (g \circ f)$.

Exer A smooth atlas on X, Y induces one on $X \times Y$.

$$A_{X \times Y} := \left\{ (U_\alpha \times V_\beta,\ \phi_\alpha \times \psi_\beta)_{(\alpha, \beta)} \right\}. //$$

$$\phi_\alpha, \psi_\beta\ C^k \Rightarrow \phi_\alpha \times \psi_\beta\ C^k.$$