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DIFFERENTIAL GEOMETRY; ITS PAST AND ITS FUTURE

by SHIING-SHEN CHERN (*)

A. Introduction.

It was almost a century ago, in 1872, that Felix Klein formulated his Erlanger Program. The idea of unifying the geometries under the group concept is simple and attractive and its applications in the derivation of different geometrical theorems from the same group-theoretic argument are usually of great elegance. This leads to the development of differential geometries of submanifolds in homogeneous (or Klein) spaces: conformal, affine, and projective differential geometries. The latter had in particular an energetic development in the twenties.

It was also about a century ago that the greatest modern differential geometer Elie Cartan was born (on April 9, 1869). Among his contributions of a basic nature are his systematic use of the exterior calculus and his clarification of the global theory of Lie groups. Fiber spaces also find their origin in Cartan's work.

Differential geometry is the study of geometry by the methods of infinitesimal calculus or analysis. Among mathematical disciplines it is probably the least understood ([1]). Many mathematicians feel there is no geometry beyond two and three dimensions. The advent into higher and even infinitely many dimensions does make the intuition unreliable and the dependence on algebra and analysis mandatory. The basis of algebra is the algebraic operations and the basis of analysis is the topological structure. I would like to surmise that the core of differential geometry is the Riemannian structure (in its broad sense).

The main object of study in differential geometry is, at least for the moment, the differentiable manifolds, structures on the manifolds (Riemannian, complex, or other), and their admissible mappings. On a manifold the coordinates are valid only locally and do not have a geometrical meaning themselves. Historically the difficulty in achieving a proper understanding of this situation must have been tremendous (I wonder whether this was part of the reason which caused Hadamard to admit his

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([1]) G. D. BIRKHOFF, « The second is a disturbing secret fear that geometry may ultimately turn out to be no more than the glittering intuitional trappings of analysis ». Fifty years of American mathematics, *Semicentennial Addresses of Amer. Math. Soc.* (1938), p. 307.

G. W. MACKEY, « Geometrical intuition, while very helpful, is not reliable and cannot be depended upon for rigorous arguments », *Lectures on the Theory of Functions of a Complex Variable*, p. 21, Van Nostrand, notes.

psychological difficulty in the mastery of Lie groups) (2). For technical purposes the Ricci calculus was a powerful tool, but it is inadequate for global problems. Global differential geometry, with the exception of a few isolated results, had to wait till algebraic topology and Lie groups have paved the way.

Global differential geometry must be considered a young field. The notion of a differentiable manifold should have been in the minds of many mathematicians, but it was H. Whitney who found in 1936 a theorem to be proved: the imbedding theorem. In the case of the richer complex structure a definition of a Riemann surface by overlapping neighborhoods was given and the theory rigorously treated by H. Weyl in his famous book " Idee der Riemannschen Fläche, Göttingen, 1913 " (3), following which Caratheodory gave the first definition of a high-dimensional complex manifold. More general pseudo-group structures were treated by Veblen and J. H. C. Whitehead in 1932 [34]. Only special cases of the general theory, such as Riemannian, conformal, complex, foliated structures, etc. have been found significant.

B. The development of some fundamental notions and tools.

Perhaps the most far-reaching achievement in differential geometry in the last thirty years lies in its foundation. Not only are the notions clearly defined, but notations are devised to treat manifolds which could be infinite-dimensional. The notations are up to now on the diversive side and are thus at an experimental stage. We believe in the survival of the fittest. Important as these foundational works are, no mathematical discipline can prosper without deeper study and simple challenging problems. We will comment briefly on a few fundamental developments in differential geometry and its related subjects, without endeavoring to make the list complete.

(1) *Lie Groups.* — It is one of the happiest incidents in the history of mathematics that the structure of Lie groups can be so thoroughly analyzed. The existence of the five exceptional simple Lie groups makes a deep study necessary and leads to a better understanding. Even so the subject has unity and is so much simpler than (say) finite groups. The quotient spaces of Lie groups give a multitude of examples of manifolds which are easy to describe. They include the classically important spaces and form a reservoir on which new conjectures can be tested.

(2) *Fiber Spaces.* — When a manifold has a differentiable structure, it can be locally linearized, giving rise to the tangent bundle and the associated tensor bundles. The first idea of a connection in a fiber bundle with a Lie group can be found in Cartan's " espaces généralisés ". It was algebraic topology which focused on the simplest problems, e. g., the problem of introducing invariants which serve to distinguish a general

(2) J. Hadamard, *Psychology of Invention in the Mathematical Field*, Princeton (1949), p. 115.

E. Cartan, in his classical « Leçons sur la géométrie des espaces de Riemann » says, « La notion générale de variété est assez difficile à définir avec précision », p. 58.

(3) Weyl's book was dedicated to Felix Klein, to whom he acknowledged for the fundamental ideas. Weyl's definition of a Riemann surface and Hausdorff's introduction of his axioms in 1914 must have made it superfluous to give formally a definition of a differentiable manifold. Chevalley's book on Lie groups (1946) exerted a great influence in the clarification of many concepts attached to it.

fiber bundle from a product bundle. Among them are the characteristic classes. Characteristic classes with real coefficients can be represented by the curvature of a connection, the simplest example being the Gauss-Bonnet formula. The bundle structure is now an integral part of differential geometry.

(3) *Variational Methods.* — The importance of the notion of measure (length, area, volume, curvature, etc.) makes the variational method a powerful and indispensable tool. The study of geodesics on a Riemannian manifold is a brilliant chapter of mathematics. It led to Morse's creation of the critical point theory whose scope extends far beyond differential geometry. Another example is the Dirichlet problem and its application to elliptic operators. Multiple integral variational problems open a vista whose terrain is still rocky. It promises, however, a fertile field of work. When a geomatrical problem involves a function, either over the given manifold or in some related functional space, it always pays to look at its critical values and the second variation at them. Much of differential geometry utilizes this idea, in its various ramifications. The importance of variational method in differential geometry can hardly be over-emphasized.

(4) *Elliptic Differential Systems.* — The geometrical properties of differential geometry are generally expressed by differential equations or inequalities. Contrary to analysis special systems with their special properties received more attention. While analysis is the main tool, geometry furnishes the variety. Differential systems on manifolds with or without boundary are the prime objects of study.

Elliptic systems occupy a central position because of their rich properties, which follow from the severe restrictions on the set of solutions. Hodge's harmonic differential forms, with their applications to Kahlerian manifolds, will remain a crucial landmark. A simple idea of Bochner relates them to curvature and leads to vanishing theorems when the curvature satisfies proper " positivity " conditions. This has remained a standard method in the establishment of such theorems, which in turn give rise to existence theorems. The indices of linear elliptic operators on a compact manifold include some of the deepest invariants of manifolds (Atiyah, Bott, Singer).

In the study of mappings an important problem consists in the analysis of the singularities. Important progress has been made recently on the singularities of differentiable mappings (Whitney, Thom, Malgrange, Mather). If the mappings are defined by elliptic differential equations, cases are known where the singularities take relatively simple form. Singularities in differential geometry remain a relatively untouched subject.

C. **Formulation of some problems with discussion of related results.**

We will attempt to discuss some areas where it is believed that fruitful researches can be carried out. The limited time at my disposal and, above all, my own limitation make it impossible for the treatment to be even remotely exhaustive. Any subject left out carries no implication that it is considered less significant.

My object is to amuse you by stating some very simple problems which have so far defied the efforts of geometers. The danger in formulating such problems is that the line distinguishing them from mathematical puzzles is thin. Personally I think there is no such line except that the " serious " problems concern with a new domain where the phenomena are not well understood and the basic concepts not well developped. Geometry and analysis on manifolds are still at this stage and will remain so for years to come. When such problems are solved, similar ones will tend toward puzzles.

Many of the problems to be given below are known. It is hoped that its collection may attract mathematicians not engaged in this field and lead to further progress.

1. Riemannian manifolds whose sectional curvatures keep a constant sign

It was known to Riemann that the local properties of a Riemannian structure are completely determined by its sectional curvature. The latter is a function $R(\sigma)$ of a two-dimensional subspace σ of the tangent space at a point x, which is equal to the gaussian curvature of the surface generated by the geodesics tangent to σ at x. Manifolds for which $R(\sigma)$ keeps a constant sign for all σ have a simple geometrical meaning. For their global study it is important to require that they are not proper open subsets of larger manifolds and, following Hopf and Rinow, it is customary to impose the stronger completeness condition: every geodesic can be indefinitely extended. In fact, without the completeness requirement the sign of the sectional curvature imposes hardly any condition on the manifold, as Gromov [21] proved that there exists on any non-compact manifold a Riemannian metric for which the range of the values of $R(\sigma)$ is any open interval on the real line.

For *complete* Riemannian manifolds M for which $R(\sigma)$ keeps the same sign the two classical theorems are:

(1) Theorem of Hadamard-Cartan. — If $R(\sigma) \leqslant 0$, the universal covering manifold of M is diffeomorphic to R^n, $n = \dim M$.

(2) Theorem of Bonnet-Myers. — If $R(\sigma) \geqslant c$ (= const) > 0, M has a diameter $\leqslant \pi/c^{1/2}$ and is therefore compact.

The case of positive curvature turns out to be more elusive. Cheeger and Gromoll [9] achieved what is essentially a structure theory of non-compact complete Riemannian manifolds M with $R(\sigma) \geqslant 0$ (all σ) by proving the following theorem. There is in M a compact totally geodesic and totally convex submanifold S_M (to be called the soul of M) without boundary such that M is diffeomorphic to the normal bundle of S_M. If the sectional curvature is strictly positive, then Gromoll and Meyer [20] proved that the soul is a point and M is diffeomorphic to R^n. In particular, M must be simply connected.

Compact Riemannian manifolds of positive curvature obviously satisfy the stronger condition $R(\sigma) \geqslant c > 0$ (all σ). By the Bonnet-Myers Theorem they are identical with the complete Riemannian manifolds with the same property. They are not necessarily simply connected, as the example of the non-euclidean elliptic space shows. So far the simply connected compact differentiable manifolds known to admit a Riemannian metric of positive curvature are the following [3]: (1) the n-sphere; (2) the complex projective space; (3) the quaternion projective space; (4) the Cayley plane; (5) two manifolds discovered by Berger, of dimensions 7 and 13 respectively.

It is very unlikely that there are no others, but nothing more is known. The following question was asked by H. Hopf:

PROBLEM I. — Does the product of two 2-dimensional spheres admit a Riemannian metric of strictly positive curvature?

More generally, it is not known whether the exotic 7-spheres, some of which are bundles of 3-spheres over 4-spheres, admit Riemannian metrics of positive curvature. The answer to the question in Problem I is probably negative. A supporting evidence is furnished by the following theorem of Berger [5]: Let M and N be compact Riemannian manifolds. Let $g(t)$ be a family of Riemannian structures on $M \times N$, such that $g(0)$ is the product structure and such that the following condition is satisfied:

$$\left.\frac{dR(\sigma)}{dt}\right|_{t=0} \geqslant 0$$

for all σ spanned at $x \in M \times N$ by a tangent vector to M and a tangent vector to N. Then

$$\left.\frac{dR(\sigma)}{dt}\right|_{t=0} = 0$$

for all such σ.

To get deeper topological properties of a manifold of positive curvature Rauch introduced the notion of *pinching*. M is said to be β-pinched if $0 < \beta \leqslant R(\sigma) \leqslant 1$ for all σ. After the pioneering work of Rauch the following are the main theorems on the topology of compact pinched Riemannian manifolds of positive curvature:

(1) (Berger-Klingenberg) [4, 25]. If a simply connected Riemannian manifold of positive curvature is β-pinched, $\beta > \frac{1}{4}$, it is homeomorphic to the n-sphere; if $\beta = \frac{1}{4}$ and it is not homeomorphic to the n-sphere, it is isometric to a symmetric space of rank 1.

(2) (Gromoll-Calabi) [19]. Let M be an n-dimensional compact simply connected Riemannian manifold of positive curvature. There exists a universal constant $\beta(n) < 1$, depending only on n, such that if M is $\beta(n)$-pinched, it is diffeomorphic to the standard n-sphere.

Similar problems can be studied on the global implications of curvature properties of complex Kählerian manifolds. A new feature is the notion of holomorphic sectional curvature, i. e., sectional curvature $R(\sigma)$, where σ is the two-dimensional real space underlying a complex line in the complex tangent space. A most attractive question is the following one formulated by Frankel:

PROBLEM II. — Let M be a compact Kählerian manifold of positive sectional curvature. Is M biholomorphically equivalent to the complex projective space?

Andreotti and Frankel [17] proved that the answer is affirmative if M is of dimension 2. The proof makes use of the classification of algebraic surfaces. Partial results were recently obtained by Kobayashi and Ochiai [26] for 3 dimensions.

2. EULER-POINCARÉ CHARACTERISTIC

Among the important topological invariants of a manifold is the Euler-Poincaré characteristic. Its role is well-known on problems such as the Lefschetz fixed-point theorem, singularities of vector fields, and indices of some elliptic operators. Geometrically it is closely related to the total curvature (*curvatura integra*) as expressed by the Gauss-Bonnet formula

$$\chi(M) = \frac{(-1)^m}{2^{3m}\pi^m m} \int_M \left(\sum_{i,j} \varepsilon_{i_1 \ldots i_{2m}} \varepsilon_{j_1 \ldots j_{2m}} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2m-1} i_{2m} j_{2m-1} j_{2m}}\right) dv \qquad (1)$$

Here M is a compact orientable Riemannian manifold of even dimension $n = 2m$, $\chi(M)$ is its Euler-Poincaré characteristic, dv is the volume element, and R_{ijkl} are the components of the curvature tensor relative to ortho-normal frames. The $\varepsilon_{i_1 \ldots i_{2m}}$ is the Kronecker symbol and is zero if $i_1, \ldots, i_{2m}$ do not form a permutation of $1, \ldots, 2m$ and is equal to $+1$ or -1 according as the permutation is even or odd.

In spite of the explicit expression for $\chi(M)$ the following has not been established:

PROBLEM III AND CONJECTURE. — If M has sectional curvatures $\geqslant 0$, then $\chi(M) \geqslant 0$. If M has sectional curvatures $\leqslant 0$, then $\chi(M) \geqslant 0$ or $\leqslant 0$, according as $n \equiv 0$ or $2 \bmod 4$.

The above statement has been proved for $n = 4$ [10] and for the case that M has constant sectional curvature. A first approach would be to study the sign of the integrand in the Gauss-Bonnet formula, a purely algebraic problem. Even this algebraic problem seems to be of great interest [33].

As with the classical Gauss-Bonnet formula the relationship is more useful for compact manifolds with boundary (in which case a boundary integral should be added to make the formula (1) valid) and the problem is more interesting for non-compact manifolds, because a deeper study of the geometry will then be necessary. We will denote by $C(M)$ the right-hand side of (1) and we shall formulate the problem:

PROBLEM IV. — Let M be a complete Riemannian manifold of even dimension. Suppose $\chi(M)$ and $C(M)$ both exist, the latter meaning that the corresponding integral converges. Find a geometrical interpretation of the difference

$$\delta(M) = \chi(M) - C(M).$$

Of course $\delta(M) = 0$ if M is compact. In two dimensions Cohn-Vossen's classical inequality says that $\delta(M) \geqslant 0$. For a class of two-dimensional manifolds Finn and A. Huber [16, 23] obtained a geometrical interpretation of $\delta(M)$, which implies that it is non-negative. Partial results on Problem IV have been obtained by E. Portnoy [30]. Perhaps the case of Kählerian manifolds has a simpler answer and should be studied first.

In a different direction Satake [31] obtained a Gauss-Bonnet formula for his V-manifolds and applied it to automorphic functions and number theory. V-manifolds are essentially manifolds with singularities of a relatively simple type.

Another problem on the Euler-Poincaré characteristic concerns compact affinely connected manifolds which are locally flat. These can be described as manifolds with a linear structure, i. e., having a covering by coordinate neighborhoods such that the coordinate transformation in overlapping neighborhoods is linear.

PROBLEM V. — Let M be a compact manifold with an affine connection which is locally flat. Is its Euler-Poincaré characteristic equal to zero?

Bensecri proved that the answer is affirmative if M is of two dimensions (For proof and generalization cf. Milnor [27]). The high-dimensional case has been investigated by L. Auslander who proved the theorem [1]: suppose the affine connection be complete and suppose that the homomorphism $h: \pi_1(M) \to GL(n, R)$ defined by the holonomy group is not an isomorphism of the fundamental group $\pi_1(M)$ onto a discrete subgroup of $GL(n, R)$. Then $\chi(M) = 0$.

It is not known whether h can imbed $\pi_1(M)$ as a discrete subgroup of $GL(n, R)$.

In spite of great developments in algebraic topology there are simple problems on the Euler-Poincaré characteristic which remain unanswered.

3. Minimal submanifolds

A minimal submanifold is an immersion $x: M^n \to X^N$ of an n-dimensional differentiable manifold M^n (or simply M) into a Riemannian manifold X^N of dimension N, which *locally* solves the Plateau problem: Every point $x \in M$ has a neighborhood U such that U is of smallest n-dimensional area compared with other n-dimensional submanifolds having the same boundary ∂U. Analytically the condition can be expressed as follows: Let D^2x be the second differential on M in the sense of Levi-Civita. Then (D^2x, ξ), where ξ is a normal vector to M at x, is a quadratic differential form, the second fundamental form relative to ξ. The differential equation to be satisfied by M is

$$\operatorname{Tr}(D^2x, \xi) = 0, \qquad \text{all } \xi. \tag{2}$$

It is a system of non-linear elliptic partial differential equations of the second order, whose number is equal to the codimension $N - n$. A minimal submanifold of dimension one is a geodesic.

We wish to study the properties of complete minimal submanifolds in a given Riemannian manifold X^N (cf. [12]). Except for geodesics the interest has so far been restricted to the case when the ambient space X^N is either the Euclidean space E^N or the unit sphere $S^N(1)$ imbedded in E^{N+1}.

For a minimal submanifold $x: M^n \to E^N$ in the Euclidean space a condition equivalent to (2) is that the coordinate functions are harmonic (relative to the induced metric). It follows that for $n > 0$ a complete minimal submanifold in E^N is non-compact.

For various reasons the case of codimension one (i. e., the minimal hypersurfaces) is the most important. Let $x_1, \ldots, x_n, z$ be the coordinates in E^{n+1}. Consider minimal hypersurfaces defined by the equation

$$z = F(x_1, \ldots, x_n) \tag{3}$$

for all $x_1, \ldots, x_n$. The following fundamental theorem generalizes the classical theorem of Bernstein and was the combined effort of de Giorgi ($n = 3$), Almgren ($n = 4$), Simons ($n \leqslant 7$), Bombieri, de Giorgi, Giusti ($n \geqslant 8$) [6, 32]. The minimal hypersurface defined by (3) must be a hyperplane for $n \leqslant 7$ and is not always a hyperplane for $n \geqslant 8$.

The main reason for this difference is the existence of absolute minimum cones in high-dimensional Euclidean space, which in turn depends on properties of compact minimal hypersurfaces in $S^n(1)$. From a general viewpoint the study of compact minimal submanifolds in $S^N(1)$ is attractive for its own sake. The first uniqueness theorem is the theorem of Almgren-Calabi [11]. If a two-sphere is immersed as a minimal surface in $S^3(1)$, it must be the equator.

By a counter-example of Hsiang [22] this theorem is not true for the next dimension. However, the following question, which can be designated as the " spherical Bernstein problem ", is unanswered:

PROBLEM VI. — Let the n-sphere be *imbedded* as a minimal hypersurface in $S^{n+1}(1)$. Is it an equator?

Two-dimensional minimal surfaces in E^N and in $S^N(1)$ have been more thoroughly studied, because of the application of complex function theory. If the surface is itself a two-sphere (hence in $S^N(1)$), severe restriction is imposed for global reason and we have the following theorem (Boruvka, do Carmo, Wallach, Chern, but mainly Calabi [8, 14]). Let the two-sphere be immersed in $S^N(1)$ as a minimal surface, such that it does not belong to an equator. Then we have: (1) N is even; (2) The total area of the surface is an integral multiple of 2π; (3) If the induced metric is of constant Gaussian curvature, it is completely determined up to motions in $S^N(1)$ and the Gaussian curvature has the value

$$K = \frac{2}{m(m+1)}, \qquad N = 2m. \tag{4}$$

(4) There are minimal two-spheres in $S^N(1)$ of non-constant Gaussian curvature; all these with a given area form a finite-dimensional space.

The immersion of the n-sphere as a minimal submanifold of $S^N(1)$ is a fascinating problem. If the induced metric has constant curvature, the immersion is given by the spherical harmonics (Takahashi). For $n > 2$ two isometric minimal immersions $S^n(a) \to S^N(1)$ are not necessarily equivalent under the motions of the ambient space (do Carmo, Wallach [15]). In view of the precise results on the two-sphere we wish to propose the following problem:

PROBLEM VII. — Consider minimal immersions $S^n \to S^N(1)$ with total area $\leqslant A$ ($=$ const) and identify those which differ by a motion of the ambient space. Is the resulting set a finite-dimensional space with some natural topology?

4. Isometric mappings

A differentiable mapping $f\colon M \to V$ of Riemannian manifolds is called *isometric* if it preserves the lengths of tangent vectors. It is therefore necessarily an immersion, and $\dim M \leqslant \dim V$. Classical differential geometry deals almost exclusively with the case that V is the Euclidean space E^N of dimension N. We believe this is the most interesting case and we will adopt this restriction in our discussion.

The first problem is that of existence. Since the fundamental tensor on a Riemannian manifold of dimension n involves $n(n+1)/2$ components, Schläfli conjectured in 1871 that every Riemannian manifold of dimension n can be locally imbedded in E^N, with $N = \frac{1}{2}n(n+1)$. This was proved by Elie Cartan in 1927 for the real analytic case. For smooth non-analytic manifolds this local isometric imbedding problem is unsolved, even for $n = 2$, unless some restriction on the metric is imposed such as the Gaussian curvature keeping a constant sign. In other words, it is not known whether any smooth two-dimensional Riemannian manifold can be locally isometrically imbedded in E^3. The answer is probably negative.

The two important global imbedding theorems are:

(1) (Weyl's Problem). A compact two-dimensional Riemannian manifold of positive Gaussian curvature can be isometrically imbedded in E^3 (as a convex surface).

(2) (Nash's Theorem [18, 28]). A compact (resp. non-compact) C^∞ Riemannian manifold of dimension n can be isometrically imbedded in E^N,

$$N = \frac{1}{2} n(3n + 11) \quad (\text{resp. } N = 2(2n + 1)(3n + 7)) \ (^4)$$

The second problem is the uniqueness of the isometric imbedding, also called rigidity, which is the problem whether an isometric immersion is determined up to a rigid motion of the ambient space E^N. Most interesting is the classical case of surfaces in E^3. Cohn-Vossen proved the rigidity of compact surfaces with Gaussian curvature $K > 0$ and the theorem was extended by Voss [35] to the case $K \geqslant 0$. Even before Cohn-Vossen, Liebmann proved that a smooth family of isometric compact convex surfaces (i. e., $K > 0$) is trivial, i. e., it consists of the surfaces obtained by the rigid motion of one member of the family. It is not known whether the same is true when the curvature condition is dropped and we believe the following problem is fundamental:

PROBLEM VIII. — Let M be a compact surface and I be the interval $-1 < t < 1$. Let $f: M \times I \to E^3$ be a differentiable mapping such that $f_t : M \to E^3$ defined by $f_t(x) = f(x, t)$, $x \in M$, $t \in I$, is an immersion for each t. Suppose that the metric ds_t^2 induced by f_t on M is independent of t. Does there exist a rigid motion $g(t)$ such that

$$f_t(x) = g(t) f_0(x), \qquad x \in M, \tag{5}$$

where the right-hand side denotes the action on f_0 by $g(t)$?

The following remarks may be relevant to the problem. Cohn-Vossen [13] proved the existence of an unstable family of compact surfaces of revolution, i. e., that the above conclusion is not true if the hypothesis that ds_t^2 is independent of t is replaced by

$$\frac{\partial}{\partial t} ds_t^2 \,|_{t=0} = \frac{\partial^2}{\partial t^2} ds_t^2 \,|_{t=0} = 0 \tag{6}$$

There are well-known examples showing that Cohn-Vossen's rigidity theorem is not true without the convexity condition $K \geqslant 0$. A generalization of the latter condition to surfaces of higher genus is the notion of *tightness*. Let $f: M \to E^3$ be an immersed surface. The tangent plane at a point x is a local (resp. global) support plane if a neighborhood of the surface at x (resp. the whole surface $f(M)$) lies at one side of it. The surface is called tightly immersed if every local support plane is a global

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(^{4}) The value for N in the case of non-compact manifolds is an improvement of NASH's value by GREENE [18].

support plane. A. D. Alexandrow proved that a real analytic tightly imbedded surface of genus one is rigid and Nirenberg [29] replaced the analyticity condition by some other conditions.

On the other hand, the notion of tightness has a meaning for polyhedral surfaces. In this case the rigidity problem asks whether the congruence of corresponding faces of two tightly imbedded polyhedral surfaces implies that they differ by a rigid motion. Cauchy's classical theorem says that this is true if the surfaces are of genus zero. But Banchoff [2] has constructed examples showing that this is untrue for surfaces of genus one. From these remarks it is anybody's guess whether the answer to the question in Problem VIII is affirmative or negative.

When M is of dimension greater than two, isometry is a strong condition and there are local rigidity theorems.

5. Holomorphic mappings

A holomorphic mapping $f: M \to V$ of complex manifolds is a continuous mapping which is locally defined by expressing the coordinates of the image point as holomorphic functions of those of the original point. The most significant example is the case when M is the complex line C and V is the complex projective line $P_1(C)$ (or the Riemann sphere), in which case the mapping is known as a meromorphic function. Much recent progress has been made in extending classical geometrical function theory to the study of holomorphic mappings.

A holomorphic mapping is called non-denegerate if the Jacobian matrix is of maximum rank at some point. For given M, V there may not exist a non-degenerate holomorphic mapping. Let B be a closed subset of V. Classically the following problem has been much studied.

Intersection or non-existence problem. Find B such that there is no non-degenerate holomorphic mapping $M \to V - B$, i. e., every non-degenerate holomorphic mapping $f: M \to V$ has the property $f(M) \cap B \neq \emptyset$.

The Picard theorem concerns the case $M = C$, $V = P_1(C)$, and B is the set of three distinct points. Clearly if the property holds for B, it holds for a subset containing B, so that a stronger theorem results from a smaller subset B. In view of the extreme importance and elegance of the Picard theorem, we wish to state the following conjecrure of Wu:

Problem and conjecture IX. — Let C_n be the n-dimensional complex number space and $P_n(C)$ the n-dimensional complex projective space. Let B be the set of $n + 2$ hyperplanes of $P_n(C)$ in general position (i. e., any $n + 1$ of them are the faces of a non-degenerate n-simplex). Then there is no non-degenerate holomorphic mapping $C_n \to P_n(C) - B$.

The Picard theorem says that this is true for $n = 1$. Wu has established this for $n \leqslant 4$. Moreover, if we set

$$\rho(n) = \begin{cases} \left(\dfrac{n}{2} + 1\right)^2 + 1, & n \text{ even} \\ \left(\dfrac{n+1}{2}\right)\left(\dfrac{n+3}{2}\right) + 1, & n \text{ odd,} \end{cases}$$

and let B' be the set of $\rho(n)$ hyperplanes in general position in $P_n(C)$, then Wu [36] proved that every holomorphic mapping $f: C_n \to P_n(C) - B'$ must reduce to a constant.

A far-reaching generalization of the Picard theory is the equi-distribution theory of Nevanlinna, which studies the frequency that a non-constant meromorphic function takes given values. In terms of vector bundles the problem can be generalized as follows [7]. Let M be a complex manifold and $p: E \to M$ a holomorphic vector bundle over M. A holomorphic mapping $s: M \to E$ is called a section if $p \cdot s =$ identity. Let W be a finite-dimensional vector space of holomorphic sections. Suppose the manifold and the bundle fulfill some convexity conditions (which are automatically satisfied in the classical case). Then we can define, to each $s(\neq 0) \in W$, a defect $\delta(s)$ satisfying the conditions: (1) $0 \leqslant \delta(s) \leqslant 1$; (2) $\delta(\lambda s) = \delta(s)$, $\lambda \in C - \{0\}$; (3) $\delta(s) = 1$ if s has no zero. The equi-distribution problem is to find an upper bound of an average of $\delta(s)$ (a sum in the case of a finite number of sections and an integral in the case of an infinite set). The problem has been studied recently by several authors.

Dual to the intersection problem is the extension problem: Given complex manifolds M, V and a closed subset $A \subset M$. When is a holomorphic mapping $M - A \to V$ the restriction of a holomorphic mapping $M \to V$?

Many extension theorems are known. In several complex variables the most famous are the Hartogs and Riemann extension theorems, which concern with the case that V is either the complex line or a bounded set of it. We wish to formulate the following problem of Hartogs type where the curvature of the image manifold enters into play:

PROBLEM X. — Let Δ be an n-ball in C_n, $n \geqslant 2$, and let V be a complete hermitian manifold of holomorphic sectional curvature $\leqslant 0$. Is it true that every holomorphic mapping of a neighborhood of the boundary $\partial\Delta$ of Δ into V extends into a holomorphic mapping of Δ into V?

It *is* known that without the curvature condition on V the assertion is not true [24]. The problem belongs to an area which might be described as " hyperbolic complex analysis ". The philosophy is that negative curvature of the receiving space limits the holomorphic mappings and allows strong theorems. In fact, a bounded holomorphic function is a mapping into a ball which has the non-euclidean hyperbolic metric.

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Department of Mathematics,
University of California
Berkeley, California 94720
(U. S. A.)

Added during proof, March 12, 1971: Problem X has been solved independently by P. Griffiths and B. Schiffman.