Batch A: Basics of smooth maps and atlases

Let $X = (X, \mathcal{A}_X)$ and $Y = (Y, \mathcal{A}_Y)$ be smooth n-manifolds. Recall that a continuous map $f : X \to Y$ is called *smooth* if

$$\psi \circ f \circ \phi^{-1}$$

is a smooth map wherever it is defined. (Here, ϕ is a chart for X, and ψ is a chart for Y.)

1. Diffeomorphisms

A function $f : X \to Y$ is called a *diffeomorphism* if (i) f is a bijection, (ii) f is smooth, and (iii) f^{-1} is also smooth.

(a) Show that the identity map id_X is a diffeomorphism.

(b) Let $\mathcal{A}_X$ and $\mathcal{A}'_X$ be two different smooth atlases for X. Show that $\text{id}_X : (X, \mathcal{A}_X) \to (X, \mathcal{A}'_X)$ is a diffeomorphism if and only if $\mathcal{A}_X \cup \mathcal{A}'_X$ is a smooth atlas for X.

(c) Show that the relation

$$\mathcal{A}_X \sim \mathcal{A}'_X \iff \mathcal{A} \cup \mathcal{A}' \text{ is a smooth atlas on } X$$

is an equivalence relation. Note that every equivalence class has a natural representative given by the *maximal* atlas. For this reason, some authors take a smooth manifold to be a pair $(X, \mathcal{A})$ where $\mathcal{A}$ is a maximal smooth atlas for X.

(d) Let $\mathbb{R}$ be the real line with the standard smooth structure. Show that the map $f : x \mapsto x^3$ is a homeomorphism. Show that it is not a diffeomorphism.

(e) The "set of all smooth manifolds" is not necessarily a set. Regardless, if somebody gives you some set of smooth manifolds, show that diffeomorphism defines an equivalence relation.

2. Smooth maps are continuous

A map $f : X \to Y$ is alternatively called smooth at $p \in X$ if there exists a chart V about $f(p)$, and a chart U about p with $f(U) \subset V$, such that

$$\psi \circ f \circ \phi^{-1}$$

is a smooth map (between open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$). Show that a smooth map f is continuous.

3. The sphere

Verify that stereographic projection from the north and south poles give a smooth atlas on S^n.