Homework Eleven

1. Metrics and good covers

A manifold is called smoothly contractible if it is smoothly homotopy equivalent to a point. A *good cover* for an n-dimensional manifold is an open cover by sets U_α such that each U_α is diffeomorphic to standard $\mathbb{R}^n$, and where every n-fold intersection

$$U_{\alpha_1} \cap \ldots \cap U_{\alpha_n}$$

is diffeomorphic to $\mathbb{R}^n$. (When dealing with non-smooth manifolds, one should replace "diffeomorphic" with "homeomorphic."

(a) Prove every smoothly contractible manifold has $H^0_{dR} = \mathbb{R}$ and $H^k_{dR} = 0$ for all $k > 0$.

(b) Prove using geodesically convex neighborhoods of Riemannian manifolds that every smooth manifold admits a good cover. To keep yourself from difficulty, you can just prove that one has a cover by open sets diffeomorphic to $\mathbb{R}^n$, where the intersections are contractible—it is tricky to show that the intersections are actually diffeomorphic to $\mathbb{R}^n$. Regardless, in future problems, you may assume that every manifold admits a good cover.

2. Poincaré Duality

(a) Let $A^\bullet$ be a cochain complex over $\mathbb{R}$. Show that the cochain complex $(A^\bullet)^\vee := \mathrm{Hom}(A^\bullet, \mathbb{R})$ given by

$$\ldots \to \mathrm{Hom}(A^k, \mathbb{R}) \to \mathrm{Hom}(A^{k-1}, \mathbb{R}) \to \ldots$$

is a cochain complex. The differential δ is given as follows: If $f \in \mathrm{Hom}(A^\bullet, \mathbb{R})$ then

$$(\delta f)a = (-1)^{|f|+1} f(da)$$

where d is the differential of $A^\bullet$. More generally, there is always a differential

$$(\delta f)(a) := d_B(f(a)) - (-1)^{|f|} f(d_A a)$$

on the cochain complex $\mathrm{Hom}(A^\bullet, B^\bullet)$. For instance, if $|f| = 0$, this measures the extent to which f commutes with d.

(b) Let M be a smooth, oriented manifold of dimension n. Consider the map
$$PD : \Omega^p_{dR}(M) \to (\Omega^{n-p}_c(M))^\vee$$
given by
$$\omega \mapsto \int_M \omega \wedge -.$$
Show this defines a map of cochain complexes
$$PD : \Omega^\bullet_{dR}(M) \to (\Omega^\bullet_c(M))^\vee.$$

(c) Show this map induces an isomorphism on cohomology
$$PD : H^p_{dR}(M) \to (H^{n-p}_c(M))^\vee$$
for $M = \mathbb{R}^n$.

(d) Suppose that U and V are open sets of a manifold M such that for each of $U, V, U \cap V$, the map PD is an isomorphism on cohomology. Show that PD is also an isomorphism for $U \cap V$. (Hint: Five Lemma from last homework. Take care with the connecting homomorphisms for the long exact sequences.)

(e) For what manifolds can you now prove that there is an isomorphism
$$PD : H^p_{dR}(M) \to (H^{n-p}_c(M))^\vee?$$
At least prove it for compact manifolds, and see what hypotheses you must impose to prove it for non-compact manifolds.

(f) Assume that the Euler characteristic of a compact manifold can also be expressed as
$$\sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{R}} H^i_{dR}(M).$$
Prove that if M is compact, odd-dimensional, and oriented, it has zero Euler characteristic.

3. The hyperbolic upper half plane

Let $M \subset \mathbb{R}^2$ be the set of all $(x, y) \in \mathbb{R}$ such that $y > 0$. We endow it with the metric
$$\frac{dx^2 + dy^2}{y^2}.$$
That is, given tangent vectors $a\partial_x + b\partial_y$ and $c\partial_x + d\partial_y$ at the point (x, y), the inner product of the two is given by
$$\frac{ac + bd}{y^2}.$$

(a) Show that any vertical line ($x = \text{constant}, y > 0$) is a geodesic.

(b) Given a 2×2 real matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

prove that the assignment

$$z \mapsto \frac{az+b}{cz+d}$$

is an isometry. Show that this defines an action of $PSL_2(\mathbb{R})$ on the upper-half-plane.

(c) Show the above action is transitive.

(d) Compute the stabilizer of the point i.

(e) Compute the curvature of the Levi-Civita connection of the above metric, at every point of M.