Homework Ten

Due Nov 17. As usual, if a problem has an asterisk on it, you do not have to hand it in. But you may assume it for the rest of the problems.

1. *The Five Lemma

The *five lemma* states the following: Consider a commutative diagram of abelian groups

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

and assume that the top and bottom rows are both exact. If every vertical map except the middle map is an isomorphism, then the middle vertical map is also an isomorphism.

Prove the five lemma. Note that you can relax the hypotheses so that the leftmost vertical map is just a surjection, and the rightmost vertical map is just an injection.

Like the previous homework about long exact sequences of cohomology groups, this will require some diagram-chasing.

2. Compactly supported deRham cohomology and Mayer-Vietoris

Let M be a smooth manifold. A differential form α on M is called *compactly supported* if $\alpha = 0$ outside of some compact subset $K \subset M$. Note that if α is compactly supported, so is $d\alpha$.

(a) Let $\Omega^k_c(M)$ denote the $\mathbb{R}$ vector space of compactly supported k-forms on M. Prove that $\Omega^*_c(M)$, with the usual deRham differential, forms a cochain complex.

We call its cohomology, $H^*_c(M)$, the *compactly supported deRham cohomology* of M.

(b) Show that if $U \subset M$ is an open subset, *extension by zero* defines a map of cochain complexes (i.e., a chain map)

$$\Omega^*_c(U) \to \Omega^*_c(M).$$

If $i : U \to M$ is the inclusion map, we let $i_* : H^*_c(U) \to H^*_c(M)$ denote the induced map on compactly supported cohomology.

(c) Let $U, V \subset M$ be open sets that cover M. Let

$$i_U : U \cap V \hookrightarrow U, \qquad i_V : U \cap V \hookrightarrow V, \qquad j_U : U \hookrightarrow M, \qquad j_V : V \hookrightarrow M$$

denote the inclusion maps. Show that there is a *Mayer-Vietoris* sequence for compactly supported deRham cohomology—that is, show that the following sequence

$$\cdots \xrightarrow{\delta} H^k_c(U \cap V) \xrightarrow{(i_U)_* \oplus -(i_V)_*} H^k_c(U) \oplus H^k_c(V) \xrightarrow{(j_U))_* \oplus (j_V)_*} H^k_c(M)$$

$$\xrightarrow{\delta} H^{k+1}_c(U \cap V) \longrightarrow \cdots$$

is exact. (Hint: Construct a map between three cochain complexes that is exact, just like in the last homework. Note also that the directions of the arrows has been reversed compared to usual deRham cohomology.)

Remark 2.1. Note that with respect to *all* smooth maps $f : M \to N$, ordinary deRham cohomology behaves *contravariantly*. However, with respect to smooth *open embeddings*, we have a theory of compactly supported cohomology that behaves *covariantly*.

3. Compactly supported deRham cohomology for $\mathbb{R}^n$

In this problem, we will compute the above cohomology groups for $\mathbb{R}^n$. The answer is non-trivial!

(a) Show that $H^k_c(\mathbb{R}^n) = 0$ for all $k > n$. (It's for an easy reason.)

(b) Show that $H^0_c(\mathbb{R}^n) = 0$ for all $n > 0$. (This is already different from usual $H^0_{dR}(\mathbb{R}^n)$.)

(c) Consider the integration map

$$H^1_c(\mathbb{R}) \to \mathbb{R}$$

given by sending a compactly supported cohomology class $[\alpha]$ to the integral $\int_{\mathbb{R}} \alpha$. Show this is well-defined. Note it is an $\mathbb{R}$-linear map of vector spaces.

(d) Show it is surjective.

(e) We will show injectivity as follows: Assume α is in the kernel of the above integration map. Writing $\alpha = f(x)dx$, consider the function

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

Show this is compactly supported, and that this function shows that α is exact.

(f) For $n \geq 2$, show that the integration map

$$H_c^n(\mathbb{R}^n) \to \mathbb{R}^n$$

is a surjection.

(g) Show that the integration map $H_c^n(\mathbb{R}^n) \to \mathbb{R}$ is injective. (Hint: You know what the ordinary deRham cohomology group is, so if $[\alpha]$ is in the kernel, then $\alpha = d\beta$ for some $(n-1)$-form β. This β may not be compactly supported, but deduce that $\int_{S^{n-1}} \beta = 0$ by Stokes's Theorem. Since S^{n-1} is smoothly homotopy equivalent to $\mathbb{R}^n$ minus a small ball, β is also exact on $\mathbb{R}^n - ball$, so we can find some γ on $\mathbb{R}^n - ball$ such that $d\gamma = \beta$. Constuct a function f so that $\beta - d(f\gamma)$ is a compactly supported form on all of $\mathbb{R}^n$ whose derivative is α.)

(h) Show that $H_c^k(\mathbb{R}^n) = 0$ for $0 < k < n$.

(i) Show that two smooth manifolds can be smoothly homotopy equivalent, but have non-equivalent compactly supported deRham cohomology.

(j) Show that if two compact smooth manifolds are smoothly homotopy eqiuvalent, then they must have isomorphic compactly supported deRham cohomology classes.

Remark 3.1. The take-away is that compactly supported cohomology detects more than the smooth homotopy equivalence class when your manifold is *non-compact*.

Recollection of some linear algebra

Let V be a real vector space. Recall that

$$T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$$

is defined to be the *tensor algebra*, or *free associative algebra* generated by V. Writing out the above formula, we have

$$T(V) \cong k \oplus V \oplus (V \otimes_{\mathbb{R}} V) \oplus (V \otimes_{\mathbb{R}} V \otimes_{\mathbb{R}} V) \oplus \ldots.$$

Given an element of the form $v_1 \otimes \ldots \otimes v_k$, and another element of the form $u_1 \otimes \ldots u_l$, their product is defined by concatenating the tensor product:

$$v_1 \otimes \ldots \otimes v_k \otimes u_1 \otimes \ldots \otimes u_l.$$

So we see that $T(V)$ is a graded associative algebra, whose degree k part is given by $V^{\otimes k}$. Note that $\mathbb{R} \cong V^{\otimes 0}$ contains the unit for this algebra.

Consider the two-sided ideal I generated by elements of the form $v \otimes v \in V^{\otimes 2}$.

Definition 3. *The exterior algebra over V is defined to be the quotient algebra $T(V)/I$.*

We will write the equivalence class of $v_1 \otimes \ldots \otimes v_k$ as

$$v_1 \wedge \ldots \wedge v_k.$$

We let $\Lambda^k V$ denote the vector subspace of $T(V)/I$ spanned by these forms, and write

$$\Lambda^\bullet V = \bigoplus_{k \geq 0} \Lambda^k V$$

for the entire quotient algebra $T(V)/I$.

Example 3.2. Let $u + u' = v$. Then

$$v \otimes v = (u + u') \otimes (u + u') = u \otimes u + u \otimes u' + u' \otimes u + u' \otimes u'.$$

The lefthand side goes to zero under the quotient map, and we end up with

$$[u \otimes u'] = -[u' \otimes u'] \in \Lambda^2 V$$

which is to say

$$u \wedge u' = -u' \wedge u.$$

Remark 3.3. By definition, the vector space $\Lambda^k V$ satisfies the following universal property: If there exists any alternating, multilinear map $f : V^{\otimes k} \to \mathbb{R}$, then f factors uniquely through a map from $\Lambda^k V$ to $\mathbb{R}$:

$$\begin{tikzcd} V^{\otimes k} \arrow[d] \arrow[dr, "f"] & \\ \Lambda^k V \arrow[r, dashed, "\exists!"] & \mathbb{R}. \end{tikzcd}$$

Our convention for $\Lambda^k V^\vee \cong (\Lambda^k V)^\vee$

Now let $V^\vee = \hom_{\mathbb{R}}(V)$ be the dual vector space. Then there is a map

$$\langle , \rangle : \Lambda^k(V^\vee) \otimes \Lambda^k V \to \mathbb{R}$$

given as follows: If we have elements of the form

$$f = f_1 \wedge \ldots \wedge f_k \in \Lambda^k V^\vee, \qquad v = v_1 \wedge \ldots \wedge v_k \in \Lambda^k V,$$

we define

$$\langle f, v \rangle := \det(f_i(v_j))$$

where on the right, we are taking the determinant of a $k \times k$ matrix whose (i, j)th entry is given by the real number $f_i(v_j)$.

The bracket above defines a homomorphism

$$\Lambda^k(V^\vee) \to \hom(\Lambda^k V, \mathbb{R})$$

where the target, by the universal property of Λ^k, is isomorphic to the vector space

$$\mathrm{Alt}(V^{\otimes k}, \mathbb{R})$$

of alternating maps.

Taking $V = T_pM$ for each $p \in M$, this shows that a differential k-form at a point p defines a map

$$(T_pM)^{\otimes k} \to \mathbb{R}$$

which is alternating. In this way, we can think of differential k-forms as specifying a way of "eating" k tangent vectors and spitting out a number, for any $p \in M$.

Remark 3.4. Another convention in use to define this map is choosing

$$\langle f, v \rangle := \frac{1}{k!} \det(f_i(v_j))$$

But we will *not* use this convention.

Example 3.5. Given a k-tuple of vector fields $Y_a \in \Gamma(T\mathbb{R}^n)$, $a = 1, \ldots, k$ and a differential k-form

$$\alpha = \sum_{I=(i_1<\ldots<i_k)} f_I dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

we get a function

$$\alpha(Y_1, \ldots, Y_k) = \sum_I f_I \det(Y_{ai_a})$$

where Y_{ab} is the bth component of the vector field Y_a. That is,

$$Y_a = \sum_b Y_{ab} \frac{\partial}{\partial x_b}.$$

Even more explicitly,

$$\alpha(Y_1, \ldots, Y_k) = \sum_I f_I \left(\sum_{\sigma \in S_k} \prod_{a=1}^{k} Y_{\sigma(a)i_a} \right)$$

4. Forms as multilinear maps

By writing things out in local coordinates, prove

$$d\alpha(Y_0,\dots,Y_k) = \sum_{i=0}^{k}(-1)^i Y_i \alpha(Y_0,\dots,\hat{Y_i},\dots,Y_k) + \sum_{0\le i<j\le k}(-1)^{i+j}\alpha([Y_i,Y_j],Y_0,\dots,\hat{Y_i},\dots,\hat{Y_j},\dots,Y_k).$$

This is just combinatorics and being careful about sign changes.

5. Line bundles are invertible

(a) Prove that for any real line bundle E, the bundle $\mathrm{Hom}(E,E)$ is trivial. Conclude that the set of isomorphism classes of line bundles on a smooth manifold E forms a group under tensor product.
(b) Prove the above statements for complex line bundles, where now $\mathrm{Hom}(E,E) = \mathrm{Hom}_{\mathbb{C}}(E,E)$ is the bundle of complex-linear maps.

In algebraic geometry, where we consider the algebraic line bundles on an algebraic space, this group is called the *Picard group* of the space.

6. *Eulerian fun

(a) Show that if $\pi : E \to M$ is a trivial vector bundle of rank $2k$, then it is orientable, and its Euler class is $0 \in H^{2k}(M)$ (regardless of orientation).
(b) Let E be a complex vector bundle, and let $E_{\mathbb{R}}$ denote the same smooth manifold, thought of as a real vector bundle of rank $2k$. Note that being a complex vector bundle means E is oriented as a real vector bundle. Show that

$$e(E_{\mathbb{R}}) = c_k(E).$$

(c) Note that $\mathbb{C}P^1 = S^2$ can be covered by two open sets, each equal to $\mathbb{C}$, with an intersection diffeomorphic to $\mathbb{C} - \{0\}$. Then the transition maps $z \mapsto 1/z$ is a holomorphic transition function, and in particular we can give the tangent bundle a complex structure. Show that this complex vector bundle is not a trivial complex vector bundle. (We've proven Euler-Gauss-Bonnet for 2-manifolds.)
(d) Likewise using the Euler class, show that the tangent bundle of S^2 is non-trivial as a real vector bundle.

The tautological bundle on projective space

Recall that $\mathbb{C}P^n$ is the space of all lines in $\mathbb{C}^{n+1}$ going through the origin. It is also described as the quotient space

$$(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^\times$$

where $\mathbb{C}^\times$ acts by
$$t(z_0, \ldots, z_n) = (tz_0, \ldots, tz_n).$$
It turns out this is a smooth manifold, which you can take for granted in this problem.

The *tautological bundle* L on $\mathbb{C}P^n$ is defined to be the submanifold
$$L \subset \mathbb{C}P^n \times \mathbb{C}^{n+1}$$
consisting of those points $(p, \vec{z})$ for which $\vec{z} \in p$. That is, $\vec{z}$ is a vector in $\mathbb{C}^{n+1}$ contained in the line $p \in \mathbb{C}P^n$.

Let $L^\perp$ denote the orthogonal complement of L inside the trivial bundle $\mathbb{C}P^n \times \mathbb{C}^{n+1}$. (For instance, by choosing the standard Hermitian metric on $\mathbb{C}^{n+1}$.

7. *Chern classes for projective spaces

(a) Show that $T\mathbb{C}P^n \cong \mathrm{Hom}_{\mathbb{C}}(L, L^\perp)$. (This isn't really possible unless we define the complex structure on $\mathbb{C}P^n$ in more detail. So you can move on if you don't figure it out.)

(b) Assuming (a), show that
$$T\mathbb{C}P^n \oplus \underline{\mathbb{C}} \cong L^\vee \oplus \ldots \oplus L^\vee.$$
This will come in handy in the rest of this problem.

(c) Let α be a smooth 2-form on $\mathbb{C}P^1$ whose integral over $\mathbb{C}P^1$ is 1 (with the orientation given from the complex structure on its tangent bundle). We denote its cohomology class by $x = [\alpha] \in H^2_{dR}(\mathbb{C}P^1)$. Show that
$$c_1(\mathbb{C}P^1) = 2x.$$
It may help to examine Problem 6.

(d) Prove
$$c_1(L^\vee) = x.$$
Using the canonical inclusion $j : \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^n$, and observing that $j^*L = L$, conclude that $c_1(L^\vee) = y$ is non-zero for $\mathbb{C}P^n$. (Here, j^*L is pulling back the tautological bundle on $\mathbb{C}P^n$, while the righthand side is the tautological bundle on $\mathbb{C}P^1$.)

(e) Assume that the Euler characteristic of $\mathbb{C}P^n$ is non-zero for all n. Explain why we have proven that there is an injective ring homomorphism
$$\mathbb{R}[t]/t^{n+1} \to H^*_{dR}(\mathbb{C}P^n).$$

(f) If you know cellular cohomology, explain why the above ring homomorphism is an isomorphism.