Homework Nine

1. Long exact sequences come from short exact sequences of cochain complexes

Recall that a cochain complex $A^\bullet$ over $\mathbb{R}$ is the data of

(1) An $\mathbb{R}$-vector space A^k for each $k \in \mathbb{Z}$, and
(2) An $\mathbb{R}$-linear map $d^k : A^k \to A^{k+1}$ for each $k \in \mathbb{Z}$, such that $d^{k+1} \circ d^k = 0$ for all k.

Recall also that a *chain map* $f : A^\bullet \to B^\bullet$ of cochain complexes is a linear map $f^k : A^k \to B^k$ for every $k \in \mathbb{Z}$ such that $d_B^k f^k = f^k d_A^k$ for all k.

Let $A^\bullet, B^\bullet, C^\bullet$ be cochain complexes over $\mathbb{R}$. Suppose one has chain maps

$$f : A^\bullet \to B^\bullet, \qquad g : B^\bullet \to C^\bullet$$

such that for each k,

$$0 \to A^k \to B^k \to C^k \to 0$$

is a short exact sequence. This means that for all $k \in \mathbb{Z}$, f^k is an injection, g^k is a surjection, and $\ker g^k = \operatorname{im} f^k$.

Prove that there exists a linear map $\delta : H^k(C) \to H^{k+1}(A)$ for every k, such that the sequence of real vector spaces

$$\ldots \xrightarrow{\delta} H^i(A) \to H^i(B) \to H^i(C) \xrightarrow{\delta} H^{i+1}(A) \to H^{i+1}(B) \to H^{i+1}(C) \xrightarrow{\delta} \ldots$$

is exact. This means that the image of every map is the kernel of the next. Note that the maps $H^i(A) \to H^i(B) \to H^i(C)$ are the maps induced by f and g.

2. The Mayer-Vietoris Sequence

Let X be a smooth manifold, and choose two open sets $U, V \subset X$ such that $U \cup V = X$.

(1) Let $j_U : U \to X$ and $j_V : V \hookrightarrow X$ be the inclusions of U and V into X.
(2) Let $i_U : U \cap V \hookrightarrow U$ and $i_V : U \cap V \hookrightarrow V$ be the inclusions of $U \cap V$ into U, and into V, respectively.

Show that for every k,

$$0 \longrightarrow \Omega^k_{dR}(X) \xrightarrow{j_U^* \oplus j_V^*} \Omega^k_{dR}(U) \oplus \Omega^k_{dR}(V) \xrightarrow{i_U^* - i_V^*} \Omega^k_{dR}(U \cap V) \longrightarrow 0$$

is a short exact sequence.

3. Basic computations of deRham cohomology

(a) Show that for all k,

$$H^k_{dR}(X \coprod Y) \cong H^k_{dR}(X) \oplus H^k_{dR}(Y).$$

(b) Compute $H^*_{dR}(S^k)$ for all k using the Mayer-Vietoris sequence. It may help to first compute it in the case $k = 0$, and use induction. Do not be afraid to use homotopy invariance of deRham cohomology, especially 2(d) of Homework 7.

4. Basic Lie groups stuff

A *smooth Lie group action* of G on a manifold X is a smooth map

$$G \times X \to X$$

such that $(gh)x = g(hx)$ and $ex = x$ for all x, all $g, h \in G$. (Here, $e \in G$ is the identity.) For the purposes of this problem, assume that G is both second countable and connected.

(a) Show that if a Lie group G acts transitively and freely on X, and if $\dim G = \dim X$, then G is diffeomorphic to X.
(b) Show that SU_n acts transitively on S^{2n-1} for all $n \geq 2$.
(c) Show that SU_2 is diffeomorphic to S^3.