Homework Eight

1. The power of metrics in understanding characteristic classes

Recall that any real vector bundle E admits a Riemannian metric, and a Levi-Civita connection.

(a) Show that if X is a skew-symmetric matrix (so $X^T = -X$) then so is X^i for any odd i.

(b) Fix a Riemannian metric on E, and let Ω be the curvature 2-form associated to the Levi-Civita connection. Prove that if i is odd, $s_i(\Omega) = 0$ *as a differential form.* (Obviously, we can then conclude that $[s_i(\Omega)] = 0 \in H^*_{dR}(M)$.) Conclude that for an odd degree invariant polynomial f, $[f(\Omega)] = 0 \in H^*_{dR}(M)$.

(c) Fix a Hermitian metric on a complex vector bundle E, and let Ω be the curvature 2-form associated to the Levi-Civita connection. By considering $\sigma_i(\Omega)$, conclude that the Chern classes can be represented by real forms, so that they lie in the *real* deRham cohomology of M.

2. Dual complex vector bundles and their Chern classes

Given a complex vector bundle E, we define its *conjugate* complex vector bundle $\overline{E}$ to be the complex vector bundle which is the *same* smooth manifold as E, and has the same projection map $\pi : \overline{E} \to M$, but for which each fiber has the conjugate complex-linear action. To be explicit, let v, u be vectors in E_p, and suppose that

$$iv = u$$

where i is the complex multiplication given to us with E. Then for $\overline{E}$, we define

$$iv := -u.$$

(a) Show that $c_k(\overline{E}) = (-1)^k c_k(E)$. (Hint: Given a connection on E, what connection does it induce on $\overline{E}$? Compare their curvatures. You probably want to be careful with your notation to differentiate between the action of i on E, and the action of i on $\overline{E}$, and you'll want to use a Hermitian metric at the end.)

(b) Further, let E^* be the complex vector bundle where the fibers are now identified with $\hom_{\mathbb{C}}(E, \mathbb{C})$. (If you like, there is a contravariant functor from $\mathsf{Vect}_{\mathbb{C}}$ to itself given by $\hom_{\mathbb{C}}(-, \mathbb{C})$. E^* is the induced vector bundle.)

Show, using a Hermitian metric, that $E^* \cong \overline{E}$ as complex vector bundles.

Note that this gives rise to the possibility that E is not isomorphic to E^*—for by part (a), the Chern classes of E and E^* may differ. This is in contrast to the real case, where a Riemannian metric always induces an $\mathbb{R}$-*linear* isomorphism $E \cong E^*$.

3. Pontrjagin classes are determined by Chern classes

Note that we have a functor

$$\mathsf{Vect}_{\mathbb{R}} \to \mathsf{Vect}_{\mathbb{C}}$$

which takes any real vector space V to the tensor product $V \otimes_{\mathbb{R}} \mathbb{C}$. This tensor product is a complex vector space because it receives an action by the complex numbers from the right, for instance. To be explicit, given a primitive element

$$v \otimes z \in V \otimes \mathbb{C},$$

we have

$$s(v \otimes z) = sv \otimes z = v \otimes sz, \qquad it(v \otimes z) = tv \otimes iz = v \otimes itz$$

for $s, t \in \mathbb{R}$. As a further explicit illustration, any element

$$\sum_i v_i \otimes z_i \in V \otimes \mathbb{C}$$

is equal to the element

$$\sum_i Re(z_i) v_i \otimes 1 + \sqrt{-1} \left(\sum_i Im(z_i) v_i \otimes i \right).$$

If we have a linear map $f : V \to V'$, we have an induced linear map $f \otimes \mathrm{id}_{\mathbb{C}} : V \otimes \mathbb{C} \to V' \otimes \mathbb{C}$.

Let E be a real vector bundle. Then one can define a complex vector bundle $E \otimes_{\mathbb{R}} \mathbb{C}$. This is called the *complexification* of E.

(a) Show that a real connection ∇ on E induces a complex connection on $E \otimes \mathbb{C}$.
(b) Show that $p_k(E)$ is equal to $(-1)^k c_{2k}(E \otimes \mathbb{C})$.
(c) Prove that if a complex vector bundle has non-zero, odd Chern classes, it cannot be the complexification of a real vector bundle.

4. Pontrjagin numbers

Let M be a $4k$-manifold for $k \geq 0$. Let f be a polynomial in the variables $x_1, \ldots, x_k$ and declare that each variable x_i has degree $4i$. Then we say that f is homogeneous of degree $4k$ if every monomial has total degree $4k$. For instance,

$$x_1^3 + x_2 x_1 + x_3$$

is a homogeneous polynomial of degree $4k$ in this convention. If $p_i \in H^{4i}(M)$ are the Pontrjagin classes of M, it makes sense to evaluate f by substituting p_i for the variable x_i, and one obtains a cohomology class of dimension $4k$. As an example, the above polynomial evaluates to

$$p_1^3 + p_2 p_1 + p_3 \in H^{12}(M).$$

Given f and an orientation on M, integrating over M thus outputs a number

$$f(p(M))[M] := \int_M f(p(M)).$$

This is called a *Pontrjagin number* of M.

Let M and M' be oriented manifolds. An *oriented cobordism* from M to M' is an oriented manifold W such that $\partial W = M \coprod -M'$, where $-M'$ is M' with the opposite orientation.

(a) Recall that the empty manifold is a manifold of every dimension. When you consider it as a manifold of dimension $4k$, show that all Pontrjagin numbers of $\emptyset$ vanish.

(b) Show that if there exists an oriented cobordism from M to M', the Pontrjagin number of M associated to a homogeneous polynomial f is equal to the Pontrajagin number of M' associated to f. This is commonly stated as: "Oriented cobordisms preserve Pontrajagin numbers."

(c) Show that if M is the boundary of a smooth, orientable $(4k+1)$-manifold, then its Pontrjagin numbers must vanish.