Homework Seven

1. Where does this come from?

Show that $\mathbb{R}^3$ with the cross product is a Lie algebra.

2. DeRham cohomology

As you may have noticed, we've been doing some simple deRham computations in homeworks just to make sure people have had a chance to work with deRham cohomology (if they haven't in a previous class).

We will say that two smooth maps $h_0, h_1 : M \to N$ are *smoothly homotopic* if there is some smooth map $H : M \times \mathbb{R} \to N$ for which $h(-, 0) = h_0$ and $h(-, 1) = h_1$. We will say that two smooth manifolds are *smoothly homotopy equivalent* if there exist maps $f : M \to N$ and $g : N \to M$ such that $f \circ g$ is smoothly homotopic to id_N, and if $g \circ f$ is smoothly homotopic to id_M.

Recall that, in class, we showed that smoothly homotopic maps induce the *same* map on deRham cohomology.

(a) Compute $H^*(\mathbb{R}^0)$ straight from the definition of deRham cohomology groups.

(b) Show that if M and N are smoothly homotopy equivalent, then we have an isomorphism $H^*(M) \cong H^*(N)$.

(c) Show that the deRham cohomology groups of $\mathbb{R}^n$ are isomorphic to the deRham cohomology groups of $\mathbb{R}^0 = pt$.

(d) Show that the deRham cohomology groups of S^n are isomorphic to those of $\mathbb{R}^{n-1} - \{0\}$.

3. Characteristic forms of curvature

Let $E \to M$ be a rank k vector bundle. Let Ω be the curvature 2-form associated to a connection on E. Let f be an invariant polynomial of degree d on $k \times k$ matrices. (This means that $f(BAB^{-1}) = f(A)$ for any $k \times k$ matrix A, and any invertible $k \times k$ matrix B.) Show that $f(\Omega)$ defines a global $2d$ form on M.

4. A connected Lie group is generated by any neighborhood of the identity

Let G be a connected Lie group, and let U be any neighborhood of the identity. (So it contains an open set containing e.) Let U^n be the set of all n-fold products of elements in U. Show

$$G = \bigcup_{n=1}^{\infty} U^n.$$

As a hint: Let $V = U \cap U^{-1}$, where U^{-1} is the set of all g^{-1} (for $g \in U$). Let H be the union of the V^n, and prove that H is both closed and open by examining its cosets.

5. Maps of Lie groups and their Lie algebras

Let G be a Lie group. Recall you've shown that $T_eG =: \mathfrak{g}$ is a Lie algebra, whose bracket is given by the Lie bracket of left-invariant vector fields. A *map of Lie groups*, or a *Lie group homomorphism*, is a smooth map $\phi : G \to H$ that is also a group homomorphism.

Show ϕ induces a map $\mathfrak{g} \to \mathfrak{h}$. Show this is a map of Lie algebras.