Homework Six

As in the previous homework, a problem given an asterirk ($*$) need not be handed in.

This homework is due in two weeks, so on Monday, October 20th. This accounts for the length.

1. Integrable and non-integrable distributions

Recall a *distribution* on N is a choice of smooth subbundle $\mathcal{H} \subset TN$. A distribution is called *integrable* if for every $p \in N$, there is an immersion $j : W \to N$ such that $Tj(TW) = \mathcal{H}|_{j(W)}$ and $p \in W$. (Note one may assume W is an injective immersion—i.e., a submanifold in the sense of a previous homework—in this definition.) Such a W is called an *integral* submanifold for the distribution.

(a) Show that if $\mathcal{H}$ is integrable, then $\Gamma(\mathcal{H})$ is closed under the Lie bracket of $\Gamma(TN)$.

The converse is called the *Frobenius theorem*, which we will eventually prove. Note that the Frobenius theorem is a theorem connecting algebra to geometry: Lie sub-algebras give rise to integrable distributions.

(b) Let $\mathcal{H} \subset T\mathbb{R}^3$ be the kernel of the differential form $dz - ydx$. Show that $\mathcal{H}$ is not integrable at the origin using the Frobenius theorem.

(c) Show that $\mathcal{H}$ is not integrable without using the Frobenius theorem.

(d) (*) Show that this distribution cannot arise from a connection on the trivial bundle over $\mathbb{R}^2 = \{(x, y)\}$.

By the way, later we will become more adept at the algebra of differential forms. As a result, we will see that any $\mathcal{H}$ defines an ideal $I \subset \Omega^*(M)$ of forms that vanish along $\mathcal{H}$. We will see that $\Gamma(\mathcal{H}) \subset \Gamma(TM)$ is a Lie subalgebra if and only if this ideal is closed under the deRham differential—so this gives a version of the Frobenius theorem. That is, one can test whether a distribution is integrable by finding certain ideals of commutative rings (the deRham algebra). This particular passage between commutative and Lie algebras is a fancy instance of a phenomenon called Koszul duality.

2. Flat connections on the real line

Let $M = \mathbb{R}$, and $E = \underline{\mathbb{R}}$ be the trivial bundle. Let $s_1 : M \to E$ be the constant section assigning $p \mapsto 1$. Let σ be an arbitrary section of $T^*M \otimes E$.

(a) (*) Show that a section of $T^*M \otimes E$ (in this problem) is the same thing as a section of T^*M.
(b) Recall from class that the assignment $\nabla : s_1 \mapsto \sigma$ defines a unique connection $\nabla : \Omega^0(E) \to \Omega^1(E)$. For the sake of notation, let us write $\sigma = gdt$. Let $s = fs_1$ be an arbitrary section of E. Write a necessary and sufficient condition on f (in terms of g) so that $\nabla s = 0$.
(c) Classify all flat connections on the trivial line bundle over $M = \mathbb{R}$.

3. Flat connections on the circle

(a) (*) Show that the cotangent bundle T^*S^1 is trivial.
(b) Classify all flat connections on the trivial line bundle on the circle.
(c) Classify all flat connections on the Mobius line bundle on the circle.

4. The space of connections

Let ∇ be a connection on $E \to M$ for an arbitrary smooth bundle E. Let $h : \Omega^0(E) \to \Omega^1(E)$ be a map that is $C^\infty(M)$-*linear*. What this means is that

$$h(fs) = fh(s)$$

for any $f \in C^\infty(M)$. Note that no such h is a connection. However,

(a) Show that $\nabla + h$ is a connection, and
(b) Show that every connection on E is obtained from ∇ by adding some choice of h to it. What you have shown is that the space of all connections is an affine space modeled on the $C^\infty(M)$-module

$$\hom_{C^\infty(M)-module}(\Omega^0(E), \Omega^1(E)).$$

5. Functoriality for connections

(a) Show that if $s_1 = s_2$ on some open set U, then $\nabla s_1 = \nabla s_2$ on U. (Hint: Bump functions.) This is what justifies us computing connections locally.
(b) Show that if we have a smooth map $f : N \to M$, and a smooth vector bundle $E \to M$ with a connection ∇, one has an induced connection ∇' on the pullback bundle f^*E. Show that this is the unique connection

on f^*E such that the diagram

$$\begin{array}{ccc} \Omega^0(E) & \xrightarrow{\nabla} & \Omega^1(E) \\ \downarrow & & \downarrow \\ \Omega^0(f^*E) & \xrightarrow{\nabla'} & \Omega^1(f^*E) \end{array}$$

commutes.

6. A non-flat connection

Let $M = \mathbb{R}^2$ and $E = \underline{\mathbb{R}}$ be the trivial bundle. Note that the usual deRham differential $d = d_{deR}$ is a connection on E. Let s be a section of E (i.e., a smooth function). We let 1 be the constant section with value 1 in the trivialization.

(a) Show that the connection

$$\nabla : s \mapsto ds \otimes 1 - ydx \otimes s, \qquad i.e., \qquad (\nabla(s))(x, y) = ds|_{(x,y)} - s(x, y)ydx|_{(x,y)}.$$

is not flat.

(b) Consider the distribution of $\mathbb{R}^3$ given as the kernel of $dz - zydx$. How is this distribution related to the connection above?

(c) Show that this distribution is not integrable using Frobenius's Theorem.

7. (*) A non-flat connection, continued

Given a smooth curve $\gamma : \mathbb{R} \to M$, a section $s : \mathbb{R} \to \gamma^*E$ is called *parallel along* γ if $\nabla'(s) = 0$. (Here, ∇' is the induced connection on γ^*E from before.) Given a parallel section s for which $s(0) = v_0 \in E_{\gamma_0}$, we say that $v_t \in E_{\gamma_t}$ is obtained by *parallel transport of* v_0 *along* γ if $v_t = s(t)$. Throughout this problem, we will use the non-flat connection on $\underline{\mathbb{R}} = E \to M = \mathbb{R}^2$ from the above problem.

(a) Let γ be the horizontal parametrized by $\gamma(t) = (t, y_0)$. Show that a section of $\gamma^*E \cong \underline{\mathbb{R}}$ is parallel if and only if can be identified as an exponential function $t \mapsto Ae^{y_0 t}$.

(b) Let γ be the vertical line parametrized by $\gamma(t) = (x_0, t)$. Show that a section of $\gamma^*E \cong \underline{\mathbb{R}}$ is parallel if and only if it is a constant function.

(c) Show that the parallel transport of some point in the fiber over the origin to another point $(x, y) \in \mathbb{R}^2$ depends on the choice of path. You may use piecewise smooth paths if it makes things easier.

8. (*) Some standard computations

This problem makes sure that we know what's going on in doing computations with differential forms.

(a) Consider the differential 1-form

$$\alpha = \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx$$

defined on the manifold $M = \mathbb{R}^2 - \{0\}$. Compute $d\alpha \in \Omega^2(M)$.

(b) Let $j : S^1 \hookrightarrow M$ be the usual inclusion. Show that $j^*\alpha$ cannot be df for any smooth function $f : S^1 \to \mathbb{R}$. (Hint: Stokes's Theorem, or the Fundamental Theorem of Calculus.)

(c) Let $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ denote a basis for $\mathbb{R}^{2n}$. Show that

$$\omega := \sum_{i=1,\ldots,n} dx_i \wedge dy_i \in \Omega^2(\mathbb{R}^{2n})$$

is a closed form.

(d) Show that ω^n is a nowhere vanishing section of $\Omega^{2n}T^*\mathbb{R}^{2n}$.

(e) Show that ω is an exact form.

(f) Let $f : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function. Exhibit conditions on f under which $f\omega$ is closed, and under which $(f\omega)^n$ is nowhere vanishing.

9. Some algebra

Definition 1. *A* commutative differential graded algebra*, or cdga, is a sequence of groups $A^k, k \in \mathbb{Z}$, together with two operations*

$$m : A^k \otimes A^l \to A^{k+l}, (a \otimes b) \mapsto ab, \qquad d : A^k \to A^{k+1}$$

such that the following holds:

(1) $ab = (-1)^{kl}ba$. *(So A is a* graded-commutative *ring).*

(2) There exists an element $1 \in A^0$ *so that* $1a = a1 = 1$ *for all* $a \in A^k$*. (So A is unital.)*

(3) $(ab)c = a(bc)$. *(So m is associative.)*

(4) $d(ab) = (da)b + (-1)^{|a|}a(db)$. *(So d satisfies the Leibniz rule; i.e., d is a derivation.)*

(5) $d^2 = 0$.

This amounts to saying that A is a *commutative algebra object* in the category of cochain complexes. Our primary example is the deRham algebra of differential forms associated to any manifold M.

Definition 2. *A* map of cdgas $f : A \to B$ *is a homomorphism* $f^k : A^k \to B^k$ *for all* $k \in \mathbb{Z}$*, such that*

$$df = fd \qquad \text{and} \qquad f(a_1a_2) = f(a_1)f(a_2).$$

(a) (*) Fix a ring R and let $A = R[x_1, \ldots, x_n]$ be a polynomial ring. Think of each element x_i as living in degree 2, so $A^0 = R$, and A^2 is the group of all homogeneous degree 1 polynomials in the x_i. Show that A is a cdga with zero differential.

(b) Given any cdga A, let $H^k(A)$ denote the kth cohomology group of A. Endow $H^*(A)$ with the zero differential $d = 0$. Show that $H^*(A)$ is a cdga. This is called the *cohomology algebra* of the cdga.

(c) Show that a map of cdgas induces a map of cdgas between their cohomology algebras.

(d) Show that any smooth map $f : M \to N$ induces a map of cdgas

$$f^* : \Omega^*(N) \to \Omega^*(M).$$

(This is *without* passing to deRham cohomology.) If you prefer, you may assume that $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$.

By the previous problem, you have shown that any smooth map $M \to N$ induces a map on their deRham cohomology algebars, $H^*(N) \to H^*(M)$.