Homework Five

If a problem is given an asterisk (*), you do not have to turn it in. If, however, you do decide to turn it in, we will grade it and you may receive extra credit.

1. Real projective space

(a) Let $\mathbb{R}P^n$ denote the set of all lines in $\mathbb{R}^{n+1}$ through the origin. For every $i = 1, \ldots, n+1$, let U_i denote the set of all lines which intersect the plane $x_i = 1$.

Given $L \in U_i$, let $\phi_i(L) \in \mathbb{R}^n \cong \{x_i = 1\}$ be the point at which L intersects the plane $x_i = 1$. That is, $\phi_i(L)$ is the point $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n, x_{n+1})$ obtained by forgetting the x_i coordinate of the intersection point with $\{x_i = 1\}$. We topologize $\mathbb{R}P^n$ by declaring that these ϕ_i are homeomorphisms, and that each U_i is open. Show that the $\{(U_i, \phi_i)\}$ define a smooth atlas for $\mathbb{R}P^n$.

(b) Exhibit a map $S^n \to \mathbb{R}P^n$ whose fibers have cardinality two. You do not need to show the following, but it may be useful to know: This is a covering map.

(c) (*) Given a line L, let x be a non-zero vector on L. For every $i, j = 1, \ldots, n+1$, we have a function

$$f_{ij} : L \mapsto \frac{x_i x_j}{\sum_{k=1}^{n+1} x_k^2}$$

which is clearly well-defined. The collection of these defines a map $\mathbb{R}P^n \to \mathbb{R}^{(n+1)^2}$. Show this defines a diffeomorphism from $\mathbb{R}P^n$ to the set of all $(n+1) \times (n+1)$ matrices which are symmetric, idempotent and trace 1.

(d) (*) Show that $\mathbb{R}P^n$ is compact.

(e) (*) Show that $\mathbb{R}P^1$ is diffeomorphic to a circle.

2. The tautological bundle

Let $\tau \subset \mathbb{R}P^n \times \mathbb{R}^{n+1}$ be the set of all pairs (L, x) where $L \in \mathbb{R}P^n$ and $x \in L$.

(a) Show that the U_i from above define trivializing neighborhoods for τ. (You can just show it for one U_i by symmetry.) Compute the transition function for one pair of unequal trivializing neighborhoods.
(b) Is τ a trivial line bundle on $\mathbb{R}P^1$?
(c) On $\mathbb{R}P^2$?
(d) On $\mathbb{R}P^n$?

3. (*) Invertible bundles

We will say that a vector bundle E is *invertible* if there is some other bundle F such that $E \otimes F \cong \underline{\mathbb{R}}$. Prove or disprove: Every smooth line bundle is invertible.

What does this say about the *structure* of the set of of isomorphism classes of line bundles?

4. Categories

(a) Let $\mathcal{C}$ be a category. Define another category $\mathcal{C}^{\mathrm{op}}$, called the *opposite category of* $\mathcal{C}$, as follows:
   (1) $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$.
   (2) $\hom_{\mathcal{C}^{\mathrm{op}}}(X, Y) := \hom_{\mathcal{C}}(Y, X)$. That is, a morphism in $\mathcal{C}$ from X to Y is treated as a morphism from Y to X in $\mathcal{C}^{\mathrm{op}}$.
   (3) Composition is induced by the composition in $\mathcal{C}$.

   Show that $\mathcal{C}^{\mathrm{op}}$ is indeed a category.[5]
(b) Fix a vector space E. Let Vect be the category of finite-dimensional vector spaces, whose morphisms are linear maps. Verify that the assignment $V \mapsto \hom(V, E)$ defines a functor $\mathsf{Vect}^{\mathrm{op}} \to \mathsf{Vect}$. (You should specify what this assignment does to morphisms.)
(c) Let $\mathcal{C}$ be a category. A morphism $f \in \hom(X, Y)$ is called an *isomorphism* if there is a morphism $g \in \hom(Y, X)$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Show that if $F : \mathcal{C} \to \mathcal{D}$ is a functor, then any isomorphism in $\mathcal{C}$ is sent to an isomorphism in $\mathcal{D}$.
(d) (*) In class, we said that part of the data of a category is a choice of identity morphism for every X. Show that this data is actually redundant–that is, show that merely asserting the existence of some $\mathrm{id}_X \in \hom(X, X)$ such that $f \circ \mathrm{id}_X = f$, $id_X \circ g = g$ (for every

[5] If you are lost, you should always think about a category with a single object. A category with a single object is the same thing as a monoid with unit. Any monoid has an "opposite" monoid, by defining right multiplication as left multiplication. An accident for a group is that the opposite group is always isomorphic to the group itself. This does not happen for monoids, and in particular, a category is almost never equivalent to its opposite!

morphism $f \in \hom(X,Y)$ and every morphism $g \in \hom(Y,X)$) is enough to specify id_X. (That is, show that id_X is unique if it exists.)[6]

(e) (*) Let $\mathcal{C}$ be the category of sets. Let W be an object of $\mathcal{C}$. Show that the assignment on objects $X \mapsto \hom(X,W)$ defines a functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Sets}$. [7]

(f) (*) If $\mathcal{C}$ is enriched over Top, show that the above functor $\hom(-,W)$ defines a functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Top}$.

(g) (*) Show that the composition of functors is again a functor, and that composition of functors is associative. Convince yourself that any category has an identity functor. In this way, one has a category of categories.

5. (*) The empty manifold's invariants

(a) What are the deRham cohomology groups of the empty manifold?

(b) For every manifold M, its deRham cohomology groups $H^*_{\mathrm{deR}}(M)$ form a (graded) unital ring. What ring is $H^*_{\mathrm{deR}}(\emptyset)$?

(c) For every smooth map $f : M \to N$, one obtains a map of rings $f^* : H^*_{\mathrm{deR}}(N) \to H^*_{\mathrm{deR}}(M)$. Note there is a unique smooth map when $M = \emptyset$. What is the induced map of deRham cohomology rings?

(d) For every manifold M, TM is a smooth manifold. What manifold is the tangent bundle of the empty manifold?

6. (*) Some categories and functors in differential topology

- Let Mfld be the category whose objects are smooth manifolds (not necessarily compact, not necessarily connected, not even necessarily of a single dimension). Morphisms are smooth maps. This can be made into a Top-enriched category in various ways, but we choose not to make any such enrichment.
- Recall that a *graded algebra over* $\mathbb{R}$ is a a pair $((A^k)_{k\in\mathbb{Z}}, m)$ where (A^k) is a collection of $\mathbb{R}$-vector spaces A^k, and m is a linear map $m : (\oplus_k A^k)\otimes(\oplus_k A^k) \to \oplus_k A^k$ such that m maps $A^k \otimes A^l$ to A^{k+l}.

[6] The proof philosophically identical to the proof that the identity element is unique in any group.

[7] This functor is called the functor *represented by* W. Let $\mathcal{C} = \mathsf{Mfld}$. By the philosophy espoused at the end of the last problem, $\hom(-,W)$ hence defines an invariant of smooth manifolds: If X does not map into W the same that way X' maps into W, then X and X' are not diffeomorphic. However, as it turns out, we can think of the *functor* $\hom(-,W)$ as an invariant of W. This functor is in fact a complete invariant of W – the functors $\hom(-,W)$ and $\hom(-,W')$ are equivalent if and only if W and W' are isomorphic in $\mathcal{C}$. This is a consequence of something called the *Yoneda Lemma.*

As part of the definition of graded algebra, m must be associative. A graded algebra is *unital* if it admits a unit $1 \in A^0$. [8]

An example is the deRham cohomology algebra of any smooth manifold.

Another example is the deRham algebra of smooth differential forms on a manifold. (I.e., what we have before taking cohomology.) This has more structure than just that of a graded algebra; we'll come back to it in another homework.

Another example is the polynomial ring in one generator, $\mathbb{R}[x]$, where A^k is the vector space of homogeneous degree k polynomials. A variant is $\mathbb{R}[y]$, where A^{2k} is the vector space of homogeneous degree k polynomials, and A^{odd} is zero. (So y is in "degree" 2.) Depending on your taste, $\mathbb{R}[y]$ is isomorphic to the deRham cohomology ring of $\mathbb{C}P^\infty$. If your taste is different, then the *power series ring* $\mathbb{R}[[y]]$ is rather isomorphic to the deRham cohomology ring of $\mathbb{C}P^\infty$.

A map between graded algebras is a linear map $f : A^k \to B^k$ for every k such that $f(m_A(a, a')) = m_B(fa, fa')$. A map between unital graded algebras further sends the unit to the unit.

We let $\mathsf{GrAlg}_{\mathbb{R}}$ be the category of unital graded algebras over $\mathbb{R}$, where morphisms are maps of graded unital algebras.

(a) Show that deRham cohomology defines a functor $H^* : \mathsf{Mfld}^{\mathrm{op}} \to \mathsf{GrAlg}_{\mathbb{R}}$. You may use facts you learned in your previous differential topology course.
(b) Explain what the above functor does to the empty manifold.
(c) Show that the "tangent bundle" defines a functor $T : \mathsf{Mfld} \to \mathsf{Mfld}$. (On objects, it assigns a manifold M to its tangent bundle TM. You must specify what the functor does to morphisms.)
(d) Explain what the above functor does to the empty manifold.
(e) Citing a previous homework, explain why the "cotangent bundle" doesn't define a functor from $\mathsf{Mfld}^{\mathrm{op}}$ to Mfld.
(f) Heuristically, an *invariant* for smooth manifolds should be an assignment of some mathematical object $I(M)$ to every smooth manifold M. We should also demand that if M is diffeomorphic to M', then $I(M)$ is somehow "equivalent" to $I(M')$. Explain why any functor out of Mfld defines an invariant for smooth manifolds.

We'll be seeing more functors out of Mfld in the homeworks to come.

[8] Some people define the graded algebra to itself be $\oplus_k A^k$. There is an issue with with this definition. For example, the ring of power series in a single variable, $\mathbb{R}[[x]]$, should be an example of a graded algebra. But $\mathbb{R}[[x]] \ncong \oplus_k \mathbb{R}$. Rather, $\mathbb{R}[[x]] \cong \prod_k \mathbb{R}$.