Homework Four

1. Locally Euclidean v. Hausdorff

For some n, find an example of a topological space which is locally homeomorphic to $\mathbb{R}^n$, but not Hausdorff.

2. Vector Bundles

Note that any vector bundle $\pi : E \to M$ admits a section called the *zero section*. That is, since the local trivializations $\pi^{-1}(U) \cong U \times \mathbb{R}^n$ are linear on the fibers, they preserve the zeroes of the vector spaces $\mathbb{R}^n$. Hence the map $x \mapsto (x, 0)$ is well-defined, and defines a section $M \to E$. In particular, it makes sense to talk about when an arbitrary section $s : M \to E$ vanishes (i.e., intersects the zero section).

(a) Let M be a smooth manifold, and $L \to M$ a line bundle. (That is, a vector bundle of rank 1.) Show that L is trivial if and only if L admits a nowhere vanishing section.
(b) Let M be a manifold of dimension n, and let $E \to M$ be a vector bundle of rank strictly greater than n. Show that E admits a nowhere vanishing section.
(c) Show that if $E \to M$ is a vector bundle admitting a nowhere vanishing section, E is isomorphic to a direct sum $\underline{\mathbb{R}} \oplus E'$, where $\underline{\mathbb{R}}$ is the trivial line bundle.

3. Some bundles on the circle

(a) Let $E' = [0,1] \times \mathbb{R}$. Define the quotient topological space E by declaring $(0,t) \sim (1,-t)$. This can be made smooth, and the natural map $E \to S^1 = [0,1]/(0 \sim 1)$ defines a vector smooth bundle over S^1 This is called the *Mobius* line bundle. Show that E is not trivial.
(b) Is $E \oplus E$ trivial?
(c) How about $E \otimes E$?

4. Tangent bundles of spheres

(a) A manifold M is called *parallelizable* if TM is trivial. Let $M = S^1, S^3$, or S^7. Show that these spheres are parallelizable. It is a deep theorem that these spheres (along with S^0) are the only parallelizable ones.
(b) If $M = S^n$ for odd n, show that TM admits a nowhere vanishing section. In particular, show that $TM \cong \underline{\mathbb{R}} \oplus E$ for some rank $n-1$ vector bundle E. Here, $\underline{\mathbb{R}}$ is the trivial rank 1 vector bundle.
(c) If $M = S^n$ where n is 3 modulo 4, show that TM is isomorphic to $\underline{\mathbb{R}^3} \oplus E'$ for some bundle E'. Here, $\underline{\mathbb{R}^3}$ is the trivial rank 3 bundle.
(d) What can you say about TS^n when n is 7 modulo 8?

In this problem, it may help to realize that the tangent bundle TS^n is isomorphic to the bundle formed by taking all pairs of vectors (x, v) in $\mathbb{R}^n$ such that $x \in S^n$ and v is tangent to S^n at x. (It isn't hard to prove.)

5. Another characterization of cotangent spaces

Let M be a smooth manifold and $p \in M$ a point. Let $Germ_p$ be the set of germs of functions at p. That is, an element of $Germ_p$ is an equivalence class $[f]$, where $f \in C^\infty(M)$. We say $f \sim g$ if for some open with $p \in U$, $f|_U = g|_U$. Let m_p denote the set of germs of functions that vanish at p. Note this is a two-sided ideal inside the ring $Germ_p$.

(a) Show that m_p is not finite-dimensional over $\mathbb{R}$.
(b) Let m_p^2 denote the ideal generated by products of elements in m_p. Show that

$$(m_p/m_p^2)^\vee \cong T_pM.$$

6. Orientability

An n-manifold M is called *orientable* if the line bundle

$$\Lambda^n(T^*M)$$

admits a nowhere-vanishing section. Show that TM is an orientable manifold for any smooth M.