Homework Three

1. Immersions, embeddings

Here, we follow terminology of Warner's text. We say a smooth map $f : M \to N$ is an *immersion* if $T_p f$ is an injection for every $p \in M$. We say that the pair (M, f) is a *submanifold* if f is further an injection as a map of sets. Finally, we say it is an *embedding* if f is further a homeomorphism onto its image. (This is equivalent to saying that the topology on M is pulled back from the subspace topology of $f(M) \subset N$.)

(a) Exhibit an example of a smooth map which is a homeomorphism, but is not an immersion.
(b) Exhibit an example of an immersion which is not a submanifold.
(c) Exhibit an example of a submanifold which is not an embedding. (You can draw a picture for this one. However you do it, explain the salient features.)

2. Lie Groups some more.

Let H and G be Lie groups. Recall that each has a Lie algebra associated to it, given by the left-invariant vector fields.

(a) Let $f : H \to G$ be a smooth map which is also a group homomorphism. Show that f induces map f_* on left-invariant vector fields, and that this is a map of Lie algebras. That is, show

$$f_*([X, Y]) = [f_*(X), f_*(Y)]$$

for any two left-invariant vector fields on H.
(b) Note that $O_n(\mathbb{R})$ is a submanifold of $GL_n(\mathbb{R})$. The tangent space $T_e GL_n(\mathbb{R})$ can be identified with the set of all $n \times n$ matrices. Characterize all $n \times n$ matrices in the tangent space $T_e O_n(\mathbb{R})$.
(c) Likewise, identify the subspace $T_e U_n \subset T_e GL_n(\mathbb{C})$. I.e., which $n \times n$ complex matrices are in the tangent space to U_n? As a hint for both problems: Any tangent vector to a manifold is a tangent vector to a curve in the manifold. If A_t is a smooth family of matrices parametrized by $\mathbb{R}$, what happens when you take the derivative of equations that look like $A_t^T A_t = I$, or $A_t^\dagger A_t = I$ with respect to t?