Homework Two

1. Functoriality

(a) Show that the assignment $X \mapsto TX$, which sends any smooth manifold to its tangent bundle, also induces a smooth map of bundles $T\phi : TX \to TY$ for any smooth map $\phi : X \to Y$. Show that $T(\phi \circ \psi) = T(\phi) \circ T(\psi)$. (By smooth map g of bundles, I mean a map which is smooth, for which $\pi \circ g = \phi \circ \pi$ for some smooth $\phi : X \to Y$, and which restricts to a linear map on fibers.)

(b) (Soft question.) In two sentences or fewer, explain why a smooth map $\phi : X \to Y$ need not induce a map $\Gamma(TX) \to \Gamma(TY)$.

(c) Show that the assignment $\Omega^1(-)$ which sends any smooth manifold to its vector space of 1-forms also assigns a linear map $\phi^* : \Omega^1(Y) \to \Omega^1(X)$ for any smooth map $\phi : X \to Y$. Show that $(\phi \circ \psi)^* = \psi^* \circ \phi^*$.

(d) (Soft question.) In two sentences or fewer, explain why a smooth map $\phi : X \to Y$ need not induce a map $T^*Y \to T^*X$.

In short, you've explained why tangent vectors like maps of bundles, but not of sections. On the other hand, forms naturally like maps of sections, but not of bundles.

2. Lie Groups

Let G be a group. Assume also that G is given a smooth atlas for which the group multiplication $G \times G \to G$ and the inverse operation $g \mapsto g^{-1}$ are both smooth. Such a G is called a *Lie group*.

For any $g \in G$, we let $L_g : G \to G$ denote left multiplication by g, so $L_g(h) = gh$. Note that since L_g is a diffeomorphism, we can push forward vector fields.

(a) We say a vector field X is *left-invariant* if $T(L_g) \circ X = X \circ L_g$. Show that left-invariant vector fields are a sub-Lie algebra of all vector fields on G. The notation here means that the diagram

$$\begin{array}{ccc} TG & \xrightarrow{TL_g} & TG \\ \uparrow{\scriptstyle X} & & \uparrow{\scriptstyle X} \\ G & \xrightarrow{L_g} & G \end{array}$$

commutes.

(b) Show that evaluation at the identity $e \in G$ induces an isomorphism between left-invariant vector fields on G, and the tangent space T_eG. This induces a non-trivial (so long as G is not abelian) Lie bracket on T_eG.

(c) (*) When $G = GL_n(\mathbb{R})$, give it the manifold structure as an open subset of $\mathbb{R}^{n^2}$, the vector space of all $n \times n$ matrices. The product and inverse operations are smooth. Show that the induced Lie algebra structure on the tangent space at the identity is the composition Lie bracket:

$$[A, B] = A \circ B - B \circ A.$$

Here, we have identified two tangent vectors at the identity with an $n \times n$ matrix, and claimed that their Lie bracket induced by left-invariant vector fields is the anticommutator of matrices.

(d) Show that the tangent bundle to a Lie group is trivial.

3. Submersions

In this problem, feel free to cite the inverse function theorem. A smooth map $f : X \to Y$ is called a *submersion* if the linear map $df : T_xX \to T_{f(x)}Y$ is a surjection for every $x \in X$. Show that if f is a submersion, for any $y \in Y$, the preimage $f^{-1}(y)$ can be given the structure of a smooth manifold such that the inclusion map $j : f^{-1}(y) \to X$ is smooth.