Math 230a Final Exam
Harvard University, Fall 2014

Instructor: Hiro Lee Tanaka

0. Read me carefully.

0.1. Due Date. Per university policy, the official due date of this exam is Sunday, December 14th, 11:59 PM. (The end of reading period.) However, there will be *no penalty* for exams handed in by Friday, December 19th, at noon. Plan out your two weeks accordingly; I hope this long window will allow you to reduce stress, and spend the time you need on this Final.

Unless your situation is exceptional, I will not accept your exam after noon on December 19th.

You may hand in exams electronically via e-mail, or leave them in my mailbox in the math department (3rd floor, right by the main math office).

0.2. Collaboration policy. For this final exam, you may *not* collaborate with other human beings, and you may not consult any human being outside of me and Robbie. (No friends, no classmates, no parents, and not even the loves of your life, unless I or Robbie happen to be one.) On the other hand, you can consult any resource mentioned, or linked to, or contained, on the course website, and also your own class notes. That's it. You may not consult any other sources.

This is a test of your understanding—your understanding of the mathematics you will learn and create during this exam—so don't cheat yourself of an honest assessment of your abilities. This has been a hard class, so I want us to make your hard work worthwhile. This should go without saying, but the university has told me that I should emphasize the obvious: I hope that you will match my desire for your learning with your integrity for your learning.

0.3. Typos and Corrections. It may be possible that this exam has typos. Check back on the website to check for updates, or e-mail me to ask whether something is a typo. As with homeworks, I will attempt to announce typos via e-mail. Please e-mail me if you suspect something is a typo.

0.4. Length. Give yourself plenty of time. You have the sixteen days for a reason. Also, though some problems have many parts, they are purposely broken into many parts to make each step of the problem simpler.

0.5. How to pass this exam. Do what you can. You do not need to successfully answer every question to obtain an A in this class. The harder problems are marked by asterisks. I think most of these questions are fun, so have fun!

1. True or False

State whether each statement below is always true, or false (i.e., not always true). **Justify your solution.**

(a) Let X and Y be smooth manifolds. If $f : X \to Y$ is a smooth homeomorphism, then it is a diffeomorphism.

(b) Let X and Y be compact smooth manifolds. If $f : X \to Y$ is a smooth homeomorphism, then it is a diffeomorphism.

(c) Let X and Y be compact, connected smooth manifolds of equal dimension. If $f : X \to Y$ is a smooth, injective immersion, then it is a diffeomorphism.

(d) If there exists a smooth function $f : X \to Y$ inducing an isomorphism of deRham cohomology groups, then X and Y are diffeomorphic.

2. DeRham cohomology of $\mathbb{R}P^n$ and $\mathbb{C}P^n$

Recall that $\mathbb{C}P^n$ is defined to be the space $\mathbb{C}^{n+1}/\mathbb{C}^\times$, and likewise, $\mathbb{R}P^n = \mathbb{R}^{n+1}/\mathbb{R}^\times$.

(a) Show that $\mathbb{C}P^n$ admits an open cover consisting of two open sets U and V, where V retracts to $\mathbb{C}P^{n-1}$, U is contractible, and their intersection retracts to a space diffeomorphic to S^{2n-1}. (Hint: Make a puncture in $\mathbb{C}P^n$ to obtain V. Let U be some tiny open ball containing the puncture. For cleanness, you might want to take the point $[1 : 0 : \ldots : 0]$ to be your puncture.)

(b) Using Mayer-Vietoris, Show that $H^k_{dR}(\mathbb{C}P^n)$ is isomorphic to $\mathbb{R}$ if k is even and $\leq 2n$, and is isomorphic to 0 for all other values of k.

(c) Using Poincaré Duality, show that the deRham cohomology *ring* of $\mathbb{C}P^n$ is isomorphic to $\mathbb{R}[x]/x^{n+1}$, where x is a generator of the degree 2 cohomology. (You'll want to recall that the Poincaré duality isomorphism is *defined* via wedge product, and induction is always a good friend.)

(d) Show that if an endomorphism of a vector space (finite-dimensional or not) satisfies the property that $f^2 = id$, then the vector space splits into two eigenspaces, of eigenvalue +1 and -1.

(e) So the antipodal map of S^n splits $\Omega^k_{dR}(S^n)$ accordingly. Show that the deRham differential respects this splitting.

(f) Let $p : S^n \to \mathbb{R}P^n$ be the 2-to-1 projection map. Show that the pullback map on deRham differential forms, p^*, induces an isomorphism from $\Omega^\bullet_{dR}(\mathbb{R}P^n)$ to the positive eigenspaces of $\Omega^\bullet_{dR}(S^n)$.

(g) Show that $H^k_{dR}(\mathbb{R}P^n)$ is isomorphic to $\mathbb{R}$ if $k = 0$ and if $k = n$ is odd. Show it is otherwise zero.

3. Künneth Theorem for DeRham cohomology

Fix a smooth manifold Y. Then for any manifold X, one can define two graded vector spaces (in fact, graded rings):

$$\bigoplus_{i+j=\bullet} H_c^i(X) \otimes_{\mathbb{R}} H_c^j(Y) \qquad \text{and} \qquad H_c^\bullet(X \times Y)$$

Here, H_c^i is compactly supported deRham cohomology (from previous homework).

(a) Show that the map sending $[\alpha] \otimes [\beta] \mapsto [\alpha \wedge \beta]$ is well-defined. Show it is in fact a homomorphism of graded rings if one gives the lefthand side the multiplication

$$[\alpha_1] \otimes [\beta_1] \cdot [\alpha_2] \otimes [\beta_2] := (-1)^{|\alpha_2||\beta_1|} [\alpha_1 \wedge \alpha_2] \otimes [\beta_1 \wedge \beta_2]$$

(b) By covering X by good covers, show the map is an isomorphism.

(c) Show that $S^2 \times S^4$ and $\mathbb{C}P^3$ have isomorphic deRham cohomology groups.

(d) Show their deRham cohomology rings are not isomorphic. (It may help to recall that smooth maps induce homomorphisms of graded rings.)

4. Chern numbers

Compute all the Chern numbers for $\mathbb{C}P^2 \times \mathbb{C}P^2$.

(That is, take the Chern classes, apply the standard invariant polynomials of appropriate degree, and integrate the result over this 4-manifold.)

5. Levi-Civita Connection

In class, we gave an explicit description of the Levi-Civita connection in terms of differential forms. (See Lecture 28.) Given a coordinate chart for $U \subset (M, g)$, define the Christoffel symbols as usual by

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k.$$

If ∇ is the Levi-Civita connection from class, prove that

$$\Gamma_{ij}^l = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) g^{kl}.$$

As usual, Einstein summation convention is being used. Here, g^{kl} is the inverse to the matrix g_{kl}.

6. Basic Riemannian geometry of immersions

Let $j : M \hookrightarrow N$ be an immersion, and let h be a Riemannian metric on N. Note that j^*TN splits into TM and a normal bundle ν—that is, the bundle of all vectors in $TN|_M$ that are orthogonal, according to h, to vectors in $Tj(TM)$. We let π denote the projection $j^*TN \to TM$.

(a) Show that pulling back h defines a Riemannian metric g on M.

(b) Let ∇^N be the Levi-Civita connection on N. Given sections s, X of TM, define
$$\nabla_X s := \pi\left((j^*\nabla^N)_X s\right).$$
That is, one pulls back the connection on TN, then applies the pull-back connection. One then projects the resulting section of j^*TN onto the component tangent to M. Show that this defines a connection on M.

(c) Show it is the Levi-Civita connection on (M, g).

(d) Show that the assignment
$$B : (X, s) \mapsto (j^*\nabla^N)_X s - \nabla_X s$$
is bilinear and symmetric in X and s. Note this is a map which takes two tangent vector fields on M and outputs a vector field on $j(M)$ normal to M.

(e) * Given $u, v \in T_pM$ that are *orthonormal*, prove that
$$K(u, v) - K^N(u, v) = \langle B(u, u), B(v, v)\rangle - |B(u, v)|^2.$$
Here, K is the sectional curvature of M, and K^N is the sectional curvature of N. This is called the *Gauss formula*.

(f) Taking $N = \mathbb{R}^3$ with the standard metric, compute the sectional curvature of a sphere of radius R in $\mathbb{R}^3$, measured with respect to a pair of orthonormal vectors tangent to the sphere.

7. Gauss's Lemma

Let (M, g) be Riemannian and fix a point $p \in M$.

(a) Fix $v \in T_pM$ for which $\exp_p$ is defined. Show that

$$\langle T\exp_p|_v(v), T\exp_p|_v(\lambda v)\rangle = \langle v, \lambda v\rangle$$

for any $\lambda \in \mathbb{R}$. Here, we are identifying a vector $v \in T_pM$ with a vector in its tangent space, $T_v(T_pM)$.

(b) Now fix a path $\alpha : (-\epsilon, \epsilon) \to T_pM$ such that $\alpha(0) = v$ and $|\alpha|$ is constant. Prove that $\alpha'(0)$ is orthogonal to v.

(c) By considering the surface

$$f(t, s) = \exp_p(t\alpha(s))$$

show that for any vector w orthogonal to v, we have

$$\langle T\exp_p|_v(v), T\exp_p|_v(w)\rangle = \langle v, w\rangle$$

(d) Show that the derivative of the exponential map *at* v preserves the inner product of v with any other vector (this is easy given the above two parts).

8. Locally minimizing property of geodesics

Let (M, g) be a Riemannian manifold and fix $p \in M$.

(a) Suppose we have a curve in T_pM of the form

$$\alpha(t) = r(t)\vec{v}(t)$$

where $r(t)$ is a non-negative function, and $\vec{v}(t)$ is a path in T_pM such that $|v(t)| = 1$ for all t. Using the Gauss Lemma, show that

$$|\frac{\partial}{\partial t} \exp \alpha(t)| \geq |r'(t)|$$

(b) Fix a number R small enough that $\exp_p$ is a diffeomorphism on the ball of tangent vectors with norm $\leq R$ about the origin of T_pM. Let B be the image of this ball under the exponential map, and suppose $\gamma : [0, 1] \to M$ is a piecewise smooth curve with image inside B and with $\gamma(0) = p$. Show that the length of γ must be greater than the length of the geodesic from $\gamma(0) = p$ to $\gamma(1)$.

(c) Show that if the length of γ equals the length of the geodesic, then the two curves must have equal image.

(d) Show that for any piecewise smooth curve $\gamma : [0, 1] \to M$—image not necessarily contained in$B$— with $\gamma(0) = p$ and $\gamma(1) \in B$,the length of γ is greater than or equal to the length of the geodesic from $\gamma(0)$ to $\gamma(1)$.

9. Hopf-Rinow Theorem

In this problem, assume the following:

THEOREM 9.1 (Hopf-Rinow). Let M be a connected Riemannian manifold. If $\exp_p$ is defined on all of T_pM, then for any $q \in M$, there is at least one geodesic from p to q such that the length of the geodesic is equal to the distance from p to q

(*Distance* is as defined in class—this is the infimum of the lengths of all piecewise smooth curves from p to q.)

(a) As a warm-up: Show that the topology defined by the distance function $d(p, q)$ is equal to the original topology of M. (Use the exponential map, and show that there are small enough open sets in both topologies.)

(b) Assume that $\exp_p$ is defined on all of T_pM. Show that any closed and bounded set of M is compact.

(c) Assume that $\exp_p$ is defined on all of T_pM. Show that any Cauchy sequence in M converges.

(d) Assume that $\exp_p$ is defined on all of T_pM. By using the locally minimizing property of geodesics and using Cauchy sequences to extend paths, show that $\exp_q$ is defined on all of T_qM for any q.

10. Expanding maps and π_1

Let X be a topological space, and x_0 a point in X. Recall that $\pi_1(X, x_0)$ is the group of all continuous paths in X that begin and end at x_0, modulo homotopy. Composition is given by concatenating paths. That is, if

$$f : [0,1] \to X, \qquad g : [0,1] \to X$$

are both continuous maps with $f(0) = f(1) = g(0) = g(1) = x_0$, then the multiplication

$$fg$$

is defined to be the homotopy class of the path given by

$$t \mapsto \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

(a) Prove that if $f : X \to Y$ is a continuous map sending x_0 to y_0, it induces a group homomorphism $\pi_1(X, x_0) \to \pi_1(Y, y_0)$.

(b) Show that if $f : X \to Y$ is a covering map sending x_0 to y_0, the group homomorphism above must be an injection.

(c) Show that if X and Y are connected, and if $f : X \to Y$ is a covering map inducing an isomorphism on π_1, then f must be a homeomorphism.

(d) Let (M, g) be a Riemannian manifold. For the purposes of this problem, let us say that a smooth map f from M to itself is *expanding* if $|Tf(v)| \geq |v|$ for all $v \in TM$. Prove that if a complete Riemannian manifold has finite fundamental group, every expanding map must be a diffeomorphism.

(e) Given an example of a compact, complete Riemannian manifold that admits an expanding map to itself which is not a diffeomorphism.

(f) * We call a smooth map *strictly expanding* if $|Tf(v)| > |v|$ for any non-zero tangent vector v. Show that if a compact Riemannian manifold admits a strictly expanding self-map $X \to X$, its universal cover must be contractible.