1. Some things you've (maybe) done before. 5 points each.

(a) If g and h are elements of a group G, show that $(gh)^{-1} = h^{-1}g^{-1}$.

$$\begin{aligned}(gh)(h^{-1}g^{-1}) &= g(hh^{-1})g^{-1} \\ &= g1g^{-1} \\ &= gg^{-1} \\ &= 1.\end{aligned}$$

Likewise,

$$\begin{aligned}(h^{-1}g^{-1})(gh) &= h^{-1}(g^{-1}g)h \\ &= h^{-1}1h \\ &= h^{-1}h \\ &= 1.\end{aligned}$$

(b) Show that the identity element of a group is unique.

If both 1 and 1' satisfy the property of being an identity, we know $1g = g$ for all g and $h1' = h$ for all h. Taking $g = 1'$ and $h = 1$, by transitivity of equality we have that $1 = 1'$.

2. You are now a Level Two Group Theorist. 5 points each.

(a) Consider the element $\sigma = (13467)$ inside S_9. What is the order of σ? (Give some reasoning.)

By *definition*, recall that a cycle is represented by the notation

$$(i\,\sigma(i)\sigma^2(i)\ldots\sigma^{|\sigma|-1}(i))$$

for some $i \in \underline{n}$ in the non-trivial orbit of σ. Hence the number of terms in the cycle notation is equivalent to the order of σ. In this case, there are five terms inside the parentheses. So the order is five.

(b) Using Lagrange's Theorem, show that any finite group with prime order $p \geq 2$ must be cyclic.

Let $g \in G$ be any element that is not the identity. The subgroup $\langle g \rangle$ must have order dividing $p = |G|$ by Lagrange's Theorem, but the only numbers dividing a prime number are 1 and p itself. On the other hand, we know that $|\langle g \rangle| \geq 2$ since this subgroup contains at least two distinct elements: 1_G and g itself. Hence $|\langle g \rangle| = p$, meaning $\langle g \rangle = G$.

3. Some (not) normal subgroups. 10 points each.

(a) Show that the subgroup generated by (123) in S_3 is normal.

This element has order 3, being a cycle of length 3. Hence it is a subgroup of index 2. (This follows from the proof of Lagrange's Theorem: $|S_3|/|\langle(123)\rangle| = 6/3 = 2$.) By homework, any subgroup of index 2 is normal. Alternatively, the group generated by (123) is A_3, so it's the kernel of a group homomorphism.

(b) Show that the subgroup generated by (123) in S_4 is *not* normal.

We know that two elements of S_n are conjugate if and only if they have the same cycle shape. Well, the cycle (124) (for instance) has the same cycle shape as (123). However,

$$\langle(123)\rangle = \{1, (123), (132)\}.$$

So (124), a conjugate of (123), is not inside $\langle(123)\rangle$. We are finished. If you want to show that (123) is a conjugate of (124) explicitly, one can compute that

$$\tau(123)\tau^{-1} = (124)$$

where $\tau = (34)$:

$$\begin{aligned}(34) \circ (123) \circ (34) &= (34) \circ (3412) \\ &= (412) \\ &= (124).\end{aligned}$$

4. Simple is simple. 10 points.

Let G be a simple group. Show that any group homomorphism from G to another group H must either be an injection, or trivial. (Trivial here means that all of G is sent to a single element of H.)

Since G is simple, its only normal subgroups are $\{1\}$ and G itself. But a kernel of a homomorphism $\phi : G \to H$ is always a normal subgroup, so any homomorphism ϕ must have kernel equal to $\{1\}$ (in which case ϕ is injective) or equal to G (in which case all of G is sent to the identity of H).

5. Some group diversity. 10 points.

For any $n \geq 3$, exhibit two groups G_n and H_n of order $n!$ which are not isomorphic. (You must explain why they are not isomorphic.)

Let $G_n = \mathbb{Z}/n!\mathbb{Z}$, and let $H = S_n$. The former is abelian, while the latter is not for $n \geq 3$, so these two groups cannot be isomorphic.

6. Extra Credit. Normal subgroups of quotients, I. 5 pts each.

Let $\phi : G \to H$ be a surjective group homomorphism.

(a) Show that if $K \subset G$ is normal, then $\phi(K) \subset H$ is normal.

We need to show that $h\phi(K)h^{-1} \subset \phi(K)$ for all $h \in H$. (We showed in class that this implies $h\phi(K)h^{-1} = \phi(K)$ for all h.)

Well, since ϕ is a surjection, there exists an element $g \in G$ for which $\phi(g) = h$. So for any $k \in K$, we have that

$$h\phi(k)h^{-1} = \phi(g)\phi(k)\phi(g)^{-1} = \phi(g)\phi(k)\phi(g^{-1}) = \phi(gkg^{-1}).$$

Since $K \subset G$ is normal, we know that $gkg^{-1} = k'$ for some $k' \in K$. Hence $\phi(gkg^{-1}) = \phi(k') \in \phi(K)$.

We've show that for every element $\phi(k) \in \phi(K)$, and for every $h \in H$, $h\phi(k)h^{-1} \in \phi(K)$. This completes the proof.

(b) Show that any normal subgroup $L \subset H$ equals $\phi(K)$ for some normal subgroup $K \subset G$.

Let

$$K = \{k \in G \text{ such that } \phi(k) \in L\}.$$

We must show that K is normal. So for any $g \in G$ and $K \in K$, we must show that $gkg^{-1} \in K$. Well,

$$\phi(gkg^{-1}) = \phi(g)\phi(k)\phi(g^{-1}) = \phi(g)\phi(k)\phi(g)^{-1} \in L$$

since L is normal in H. Hence $gkg^{-1} \in K$. This completes the proof.

7. Extra Credit. Normal subgroups of quotients, II. 10 pts.

Let $\phi : G \to H$ be a surjective group homomorphism as before. Show that there is a bijection between the set

$$\{K \subset G \text{ such that } K \text{ is a normal subgroup containing } \ker \phi\}$$

and the set

$$\{L \subset H \text{ such that } L \text{ is a normal subgroup of } H.\}.$$

For sanity, let us call the first set $\mathcal{K}$, and the second set $\mathcal{L}$. In (b) of the last problem we exhibited a function

$$j : \mathcal{L} \to \mathcal{K}$$

by sending

$$L \mapsto j(L) = \{k \in G \text{ s.t. } \phi(k) \in L\}.$$

(In the previous problem, $j(L)$ was called K.) Note that $j(L)$ contains the kernel of ϕ, since it in particular contains all k that map to $1 \in L$. We must show that j is a bijection.

On the other hand, by (a) of the last problem, we have a function

$$h : \mathcal{K} \to \mathcal{L}$$

sending $K \mapsto \phi(K)$. Let us show that j and h are inverse to each other. That $h \circ j = \mathrm{id}_{\mathcal{L}}$ is obvious—for $h(j(L))$ is the image of all elements that map to L, i.e., L. Now we must prove that $j(h(K)) = K$.

For simplicity of notation, let $h(K) = L$. If $g \in j(L)$, $\phi(g) \in L$ by definition of $j(L)$. On the other hand, $\phi(K) = L$, so there is some $k \in K$ such that $\phi(g) = \phi(k)$. This means that $\phi(gk^{-1}) = 1_L$, so $gk^{-1} = x$ is in the kernel of ϕ. But both K and $j(L)$ contain the kernel of ϕ, so by closure of subgroups, we must have that $g = xk \in K$, and $k = gx^{-1} \in j(L)$. This shows $j(h(K)) \subset K$ and $K \subset j(h(K))$. We are finished.

8. Extra Credit. 5 points each.

(a) State Mordell's Theorem. (You may use the word "nice" without defining it. You do not need to prove anything.)

Let $f(x)$ be a nice cubic polynomial with rational coefficients. Then $\mathbb{E}(\mathbb{Q})$ is finitely generated.

(b) What is the fundamental group of $\mathbb{R}^3$ with two disjoint lines removed? (You do not need to prove it; you may just state the answer.)

The free group on two generators.