Practice problems for Midterm 1

The midterm will take place Wednesday, October 15th, right after Columbus Day weekend.

1. Definitions

Know the definitions of the following. [2]

(a) Group
(b) Subgroup
(c) Order of an element
(d) Order of a group
(e) Group homomorphism
(f) Group isomorphism
(g) Kernel
(h) Image
(i) Normal subgroup
(j) Conjugation by h
(k) Conjugacy class of an element of a group
(l) $\mathrm{Aut}_{Set}(X)$
(m) Group action on a set X.
(n) Orbit
(o) Orbit space
(p) Index of a subgroup (be precise!)
(q) Stabilizer
(r) Center of a group
(s) Abelian group
(t) When H is normal, the group operation on G/H.

[2]If you are like me, you may also despise the memorization aspect of math. I've always disliked how tests force you to memorize. But we need a common and efficient language to speak of complex ideas, and this section is meant to make sure that you can communicate using the language on which English-speaking mathematicians have come to agree. Moreover, I hope that some of these ideas become intuitive enough that you will even be able to *guess* the correct definition, based on how we have come to know these words.

(u) $\mathbb{Z}/n\mathbb{Z}$
(v) S_n
(w) A_n
(x) Simple group

2

(a) Prove that the kernel of any group homomorphism is a normal subgroup.

3

(a) Show that if two cyclic groups have the same order (finite or not), they must be isomorphic.

4

(a) Exhibit an explicit element τ showing that (123)(45) and (253)(16) are conjugate in S_6.
(b) Show that S_n has at least n distinct subgroups of order $(n-1)!$.
(c) Write down every subgroup of S_3 explicitly. That is, what are the subsets of S_3 that are subgroups? When you write elements of S_3, use cycle notation.

5

(a) If S is a finite set, show that the free group on S is finitely generated.
(b) Prove that any finite group is finitely generated.

6

(a) Show that $\mathbb{Z}$ is not simple.
(b) Show that S_3 is not simple.
(c) Show that $\mathbb{Z}/12\mathbb{Z}$ is not simple.
(d) Show that A_4 is not simple.

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(a) Let H be the subgroup of S_5 generated by (13)(245). Write down every element of H.
(b) Compute the index of H inside S_5.

8

(a) State the first isomorphism theorem.
(b) State Lagrange's theorem.

9

Show by example that a subgroup of a simple group need not be simple. (You may assume that A_5 is simple.)

10

Recall that the Hamiltonians, or the quaternions, is the name for $\mathbb{R}^4$ equipped with the following operation: If $(s, \vec{u})$ and $(t, \vec{v}) \in \mathbb{R} \times \mathbb{R}^3 \cong \mathbb{R}^4$ are elements, we define

$$(s, \vec{u}) \cdot (t, \vec{v}) := (ts - \vec{u} \cdot \vec{v}, \qquad t\vec{u} + s\vec{v} + \vec{u} \times \vec{v}).$$

Here, $\vec{u} \cdot \vec{v}$ indicates the dot product of $\vec{u}$ with $\vec{v}$. In the last coordinate, $\vec{u} \times \vec{v}$ is the cross product in $\mathbb{R}^3$.

Let S^3 denote those elements $(s, \vec{u}) \in \mathbb{R}^4$ for which $s^2 + |\vec{u}|^2 = 1$. Show that S^3 is a group under the above multiplication. Show that S^3 is not an abelian group.

11. Short exact sequences

Show that following sequences do not split:

(a) $\mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ for $n \neq 0, \pm 1$.

(b) $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ where $\phi([0]) = [0]$ and $\phi([1]) = [2]$.

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If n and m are relatively prime (meaning they share no common divisors aside from 1) show that $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/(nm)\mathbb{Z}$.