Fix alg. closed field F.

<u>Last time</u> If $A: V \to V$ is a linear transformation and V is finite-dimensional, $\exists$ basis for V s.t. A is represented by the matrix (block-diagonal)

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_\ell \end{pmatrix}$$

where

$$A_i = \underbrace{\begin{pmatrix} \alpha_i & 1 & & 0 \\ & \alpha_i & 1 & \\ & & \alpha_i & \ddots \\ 0 & & & \alpha_i \end{pmatrix}}_{d_i}.$$

α_i along diagonal
1 above each α_i

We saw this by recognizing that the $F[t]$-module

$$F[t]/(t-\alpha_i)^{d_i}$$

has basis

$$\overline{(t-\alpha_i)^{d_i-1}}, \dots, \overline{(t-\alpha_i)}, \overline{1}$$

and "multiplication by t" is represented by A_i above.

We also defined

Defn The characteristic polynomial of a matrix $A \in M_{k\times k}(F)$ is the polynomial

$$\det(tI - A) \in F[t].$$

and I claimed

Thm (Cayley-Hamilton) Any matrix A satisfies its characteristic polynomial.

Rmk The theorem holds even when F is NOT algebraically closed!

Ex If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its charac. polyn. is $t^2 - t(a+d) + (ad-bc)$.
The theorem states that

$$A^2 - A(a+d) + I(ad-bc) = 0 \in M_{2\times 2}(F).$$

Ex Alternatively, if $A \in M_{k\times k}(F)$, then its charac. polyn. is of the form

$$t^k + b_{k-1}t^{k-1} + \cdots + b_1 t + b_0.$$

The theorem says that $\forall\, \vec{v} \in F^k$, we have

$$A^k \vec{v} + b_{k-1}A^{k-1}\vec{v} + \cdots + b_1 A\vec{v} + b_0\vec{v} = 0.$$

<u>Prop</u>: Let $A \in M_{k\times k}(F)$. A is invertible iff the columns of A form a basis.

Let T_A be the linear transformation $F^k \to F^k$ given by A.

<u>Pf</u>: Need to show that T_A is invertible — ie, that T_A is an injection and a surjection. Well,

$$T_A(\vec{e}_i) = \vec{v}_i \quad \text{if } v_i \text{ are columns of } A.$$

$$\begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \\ \vec{v}_i \\ \\ \end{pmatrix}$$

So
$$T_A\left(\sum b_i \vec{e}_i\right) = \vec{0} \iff \sum b_i \vec{v}_i = 0 \iff b_i = 0.$$

Since $\ker T_A = \{0\}$, and T_A is a linear map btwn F^k and F^k (which have same dim) T_A is invertible

Prop'n Let A' be a matrix for T_A in some basis $v_1, \ldots, v_k$. Let $B = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{pmatrix}$. Then

$$A' = BAB^{-1}.$$

Pf

$$\begin{aligned} \cancel{A'(\vec{v}_i)} \quad & BAB^{-1}(\vec{v}_i) \\ &= BA\vec{e}_i \\ &= B(\textstyle\sum A_{ji}\vec{e}_j) \\ &= \sum A_{ji}\vec{v}_j. \\ &= A'(\vec{v}_i). \quad // \end{aligned}$$

Prop'n If B is invertible and $A' = BAB^{-1}$, then

$$\det(tI - A') = \det(tI - A) \in F[t].$$

Pf In general, ~~since~~

$$\det(BCB^{-1}) = \det B \det C \det B^{-1}$$
$$= \det B \det B^{-1} \det C$$
$$= \det BB^{-1} \det C$$
$$= \det I \det C$$
$$= \det C$$

So

$$\det(B(tI-A)B^{-1}) = \det(tI-A)$$

while

$$\det(B(tI-A)B^{-1}) = \det(BtIB^{-1} - BAB^{-1})$$
$$= \det(tI - BAB^{-1}). \quad //$$

So to compute charac polyn of arbitrary $A \in M_{k\times k}(F)$? Well, if F is alg. closed, we know

$$A = BA'B^{-1}$$

where

$$A' = \begin{pmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_e \end{pmatrix}$$

is block diagonal, with

$$A_i = \begin{pmatrix} \alpha_i & 1 & 0 & & O \\ 0 & \alpha_i & 1 & & \\ 0 & 0 & \alpha_i & \ddots & \\ & & & \ddots & 1 \\ O & & & & \alpha_i \end{pmatrix}, \quad \alpha_i \in F$$

(α_i along diagonal, 1 right above diagonal)

So we need only see

$$\det(tI - A') = \det \begin{pmatrix} t-\alpha_1 & -1 & & O \\ 0 & t-\alpha_1 & \ddots & \\ & & \ddots & -1 \\ O & & & t-\alpha_e \end{pmatrix}$$

$$= (t-\alpha_1)^{n_1} \cdot (t-\alpha_2)^{n_2} \cdot \ldots \cdot (t-\alpha_e)^{n_e}$$

(since $tI - A'$ is upper-triangular, its det is $\prod$ of its diagonal entries.)

To show that

$$(A-\alpha_1)^{n_1} \circ \cdots \circ (A-\alpha_\ell)^{n_\ell} = 0$$

we need only show that $\forall \vec{v}$, we have

$$(A-\alpha_1)^{n_1} \circ \cdots \circ (A-\alpha_\ell)^{n_\ell} \vec{v} = 0.$$

Well, choose a basis

$$\vec{v}_{1,n_1} = \bar{1}, \ldots, \vec{v}_{1,1} = \overline{(t-\alpha_1)^{n_1-1}} \in F[t]/(t-\alpha_1)^{n_1}$$

$$\vdots$$

$$\vec{v}_{\ell,n_\ell} = \bar{1}, \ldots, \vec{v}_{\ell,1} = \overline{(t-\alpha_\ell)^{n_\ell-1}} \in F[t]/(t-\alpha_\ell)^{n_\ell}.$$

Then $\{\vec{v}_{i,n_j}\}$ form a basis for the vector space on which A acts.

Moreover,

$$(A-\alpha_1)^{n_1} \cdots (A-\alpha_i)^{n_i} \cdots (A-\alpha_\ell)^{n_\ell} \vec{v}_{i,j}$$

$$= (A-\alpha_1)^{n_1} \circ \cdots \circ (A-\alpha_\ell)^{n_\ell} (A-\alpha_i)^{n_i} \vec{v}_{i,j}$$

$$= (A-\alpha_1)^{n_1} \circ \cdots \circ (A-\alpha_\ell)^{n_\ell} \vec{0}$$

$$= \vec{0}.$$

We know (by choice of basis),

$$A\vec{v}_{i,j} = \vec{v}_{i,j-1} + \alpha_i \vec{v}_{i,j}.$$

OTOH,

$$A\vec{v}_{i,1} = \alpha \vec{v}_{i,1}.$$

So

$$(A-\alpha_i)\vec{v}_{i,j} = \begin{cases} \vec{v}_{i,j-1} & j>1 \\ 0 & j=1 \end{cases}$$

So we've shown that if F is alg. closed, any $A \in M_{k\times k}(F)$ satisfies $\det(tI-A)$. But if F isn't? Well, let $\overline{F}$ be an alg closed field into which F injects. Then we have injections

$$M_{k\times k}(F) \hookrightarrow M_{k\times k}(\overline{F})$$

Think of any matrix w/ entries in F as one w/ entries in $\overline{F}$.

and

$$F[t] \hookrightarrow \overline{F}[t].$$

Likewise — think of any polynomial w/ F coefficients as having $\overline{F}$ coefficients.

Putting it all together,

$$\begin{tikzcd}
A \arrow[r, mapsto] \arrow[ddd, mapsto, bend right] & (A, \det(tI-A)) \arrow[r, mapsto, "\text{evaluate}"] & ? \\
M_{k\times k}(F) \arrow[r] \arrow[d, hook] & M_{k\times k}(F)\times F[t] \arrow[r] \arrow[d, hook] & M_{k\times k}(F) \arrow[d, hook] \\
M_{k\times k}(\overline{F}) \arrow[r] & M_{k\times k}(\overline{F})\times \overline{F}[t] \arrow[r, "\text{evaluate}"] & M_{k\times k}(\overline{F}) \\
A \arrow[r, mapsto] & (A, \det(tI-A)) \arrow[r, mapsto] & 0
\end{tikzcd}$$

is commutative. So $? \mapsto 0$ implies $?$ must have been zero. Most succinctly: Plugging in a matrix into its characteristic polynomial yields the same result whether we think of the matrix as having F or $\overline{F}$ entries. //