Mon, Nov 24, 2014

Last Time: M an $F[t]$-module $\Rightarrow$ M an F vector space, and $t \in F[t]$ determines F-linear ~~tra~~ map

$$A : M \longrightarrow M$$
$$\vec{v} \longmapsto t\vec{v} .$$

Defn Let M be a F vector space, and fix a basis $\vec{v}_1, \ldots, \vec{v}_k$ for M. (Assume M fin.-dim.) Then given any linear transformation

$$A : M \longrightarrow M.$$

the matrix for A with respect to $\vec{v}_1, \ldots, \vec{v}_k$ is the matrix for which

$$A\vec{v}_i = \sum_{j=1}^{k} A_{ji} \cdot \vec{v}_j .$$

Ex

$$(A) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$ Then $A\vec{v}_1 = A_{11}\vec{v}_1 + A_{21}\vec{v}_2 + A_{31}\vec{v}_3$.

Exer: Write out the matrix for multiplication by t on each of the following $F[t]$-modules wrt indicated bases:

(1) $M = F[t]/(t)$ $\qquad \vec{v}_1 = \bar{1}$

(2) $M = F[t]/(t-\alpha)$ $\qquad \vec{v}_1 = \bar{1}$

(3) $M = F[t]/(t-\alpha)^2$ $\qquad \vec{v}_2 = \bar{1}, \quad \vec{v}_1 = \overline{t-\alpha}$

(4) $M = F[t]/(t-\alpha)^3$ $\qquad \vec{v}_3 = \bar{1}, \quad \vec{v}_2 = \overline{t-\alpha}, \quad \vec{v}_1 = \overline{(t-\alpha)^2}$

Soln In general, the action of $F[t]$ on $F[t]/I$ is $f \cdot \bar{g} = \overline{fg}$.

(2): $t \cdot \bar{1} = \overline{t1} = \overline{t-\alpha} + \bar{\alpha} = \bar{\alpha} = \alpha \bar{1}$.

So t sends $\bar{1}$ to $\alpha\bar{1}$ i.e.,

$$(A) = (\alpha).$$

(3) $t \cdot \bar{1} = \overline{t1} = \overline{t-\alpha} + \bar{\alpha} = \vec{v}_1 + \alpha\vec{v}_2$. So $(A) = \begin{pmatrix} ? & 1 \\ \cdot & \alpha \end{pmatrix}$

$t \cdot \overline{(t-\alpha)} = (t-\alpha)\cdot\overline{(t-\alpha)} + \alpha\cdot\overline{(t-\alpha)} = \overline{(t-\alpha)}^2 + \alpha\overline{(t-\alpha)}$

$= 0 + \alpha\vec{v}_1$. $\quad \Rightarrow A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$.

Prop. In basis ~~$\overline{1}, \overline{(t-\alpha)}, \ldots$~~ for $M = F[t]/(t-\alpha)^n$ given by

$$\overline{(t-\alpha)^{n-1}},\ \overline{(t-\alpha)^{n-2}},\ \ldots,\ \overline{(t-\alpha)^{1}},\ \overline{1},$$

the linear transformation

$$M \longrightarrow M$$
$$\vec{v} \longmapsto t\vec{v}$$

has the matrix

$$\begin{pmatrix} \alpha & 1 & 0 & & 0 & 0 \\ 0 & \alpha & 1 & & 0 & 0 \\ 0 & 0 & \alpha & & & \\ 0 & 0 & 0 & \ddots & \alpha & 1 \\ & & & & 0 & \alpha \end{pmatrix}$$

— α along diagonal, 1 directly above each α except the topmost α.

$$t V_i = t\,\overline{(t-\alpha)^{n-i}} = \left((t-\alpha)+\alpha\right) \overline{t-\alpha}^{\,n-i}$$
$$= (t-\alpha)\overline{(t-\alpha)^{n-i}} + \alpha\,\overline{(t-\alpha)^{n-i}}$$
$$= \overline{(t-\alpha)^{n-i+1}} + \alpha\vec{V_i}$$
$$= \begin{cases} \vec{V}_{i-1} + \alpha\vec{V_i} & i \le n-1 \\ \alpha\vec{V}_n & i = n \end{cases} \quad //$$

Rmk If M, N are fin-dim $F[t]$-modules and $\exists$ bases $v_1, \dots, v_m$ for M
$w_1, \dots, w_n$ for N

s.t. "multiplication by t" is given by a matrix A for M
matrix B for N,

then on $M \oplus N$, t acts by the matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$ — ie, block diagonal.

~~Pf~~ Knowing any $F[t]$-module is $\cong$ to $\bigoplus_i F[t]/(p_i^{n_i})$, we have:

Cor Let $T: F^n \to F^n$ be a F-linear transformation, F alg. closed. Then $\exists$ basis for F^n such that

$$T = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & & 0 \\ 0 & 0 & \ddots & A_e \end{pmatrix} \qquad (p_i = t - \alpha_i).$$

where $A_1 = \begin{pmatrix} \alpha_1 & 1 & 0 \\ 0 & \ddots & 1 \\ & & \alpha_1 \end{pmatrix}$, $A_2 = \overbrace{\begin{pmatrix} \alpha_2 & 1 & 0 \\ 0 & \alpha_2 \ddots & 1 \\ & & \alpha_2 \end{pmatrix}}^{n_2}$, $\dots$, $A_e = \overbrace{\begin{pmatrix} \alpha_e & 1 & 0 \\ 0 & \alpha_e \ddots & 1 \\ & & \alpha_e \end{pmatrix}}^{n_e}$.

Defn This is called Jordan normal form of T.

Defn Characteristic polynomial:

$$\det(tI - A) \in F[t].$$

Ex If $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ then

$$tI - A = \begin{pmatrix} t - A_{11} & A_{12} \\ A_{21} & t - A_{22} \end{pmatrix}$$

So

$$\det(tI - A) = t^2 - (A_{11} + A_{22})t + (A_{11}A_{22} - A_{12}A_{21}).$$

Rmk If A is $k \times k$ matrix, the i^{th} coefficient of charac. polyn. is an invariant # of A that remains unchanged under conjugation.

$$\det(B(tI-A)B^{-1}) = \det BB^{-1} \det(tI - A) = \det(tI - A).$$

Defn Minimal polynomial: Any $A \in M_{k \times k}(F)$ determines

$$f : F[t] \to M_{k \times k}(F),$$

Since $F[t]$ is a PID,

$$\ker(f) = (p), \quad p \in F[t].$$

Taking p to be unique one s.t. • $p = t^d + a_1 t^{d-1} + \cdots$

• $\ker(f) = (p)$,

we say p is minimal polynomial of A.

<u>Thm</u> (Cayley-Hamilton)
Any matrix A satisfies its characteristic polynomial.

<u>Pf</u> In basis given by above,

$$\det(tI-A) = \prod_{i=1}^{\ell} (t-\alpha_i)^{n_i}.$$

~~Moreover,~~

So NTS

$$\prod_{i=1}^{\ell} (A-\alpha_i I)^{n_i} = 0.$$

But $\mathbb{1}_1, \ldots, \mathbb{1}_\ell$ generate V as a module,

and $(A-\alpha_i I)^{n_i} \mathbb{1}_j = 0$. (Since $(A-\alpha_i I)^{n_i-1}(\mathbb{1}_i)$ is an eigenvector) //

<u>Cor</u> The minimal polynomial of A divides its characteristic polynomial.