FRI, Nov 14, 2014

How should you think of ideals?

Algebraically: A subgroup closed under scaling.

$I \subset R$ s.t.

- $rx \in I \quad \forall\, x \in I,\ r \in R$
- $(I, +) \subset (R, +)$ subgroup.

You might find this unenlightening. So:

Geometrically: Let $R = \{$ continuous functions from a space X to $\mathbb{R}\}$.

ex: $X = \mathbb{R}, \mathbb{R}^2, \ldots$

ex: X a subset of $\mathbb{R}^n$ like $S^2 \subset \mathbb{R}^3$.

This R is a ring because:

- sum of continuous fns is continuous,
- product of cont. fxns is cts,

- the zero fxn is the additive identity:

$$(0+f)(x) = 0(x) + f(x) = f(x)$$

so $0+f = f$

(likewise, $f = f+0$)

- the constant function $1: x \mapsto 1_{\mathbb{R}}$ is the multiplicative unit:

$$(1 \cdot f)(x) = 1(x) \cdot f(x) = 1_{\mathbb{R}} \cdot f(x) = f(x)$$

so $1f = f$.

(likewise, $f \cdot 1 = f$).

- $-f$ sends $x \mapsto -f(x)$, is additive inverse

- $f \cdot (g+h) : x \mapsto f(x) \left((g+h)(x) \right)$

$$= f(x) \left(g(x) + h(x) \right)$$
$$= f(x) g(x) + f(x) h(x)$$

so $f(g+h) = fg + fh$.

- associativity can also be checked easily.

Okay, so $R = \{\text{Continuous functions}\}$.

Let $Y \subset X$ be a subset, and define

$$I_Y = \{\text{functions } f \in R \text{ such that } f(y) = 0 \;\; \forall\, y \in Y\}$$

i.e., $I_Y = \{\text{fxns vanishing on } Y.\}$.

<u>Prop</u> $I_Y \subset R$ is an ideal.

<u>Pf</u> Let $f_1, f_2 \in I_Y$. Then $\forall\, y \in Y$,

$$(f_1 + f_2)(y) = f_1(y) + f_2(y) = 0 + 0 = 0$$

so $f_1 + f_2 \in I_Y$. Likewise,

$$(-f_1)(y) = -f_1(y) = -0 = 0.$$

So $-f_1 \in I_Y$. (Note this implies $0 \in I_Y$. But more straightforwardly, $0(y) = 0 \;\forall\, y \in Y$, so $0 \in I_Y$.)

So $I_Y \subset R$ is a subgrp. We just need to check it's closed under scaling by R.

Given $g \in R$, $f \in I_Y$,

$$(\underbrace{g \cdot f}_{\mathbb{R}})(y) = \underbrace{g(y) \cdot f(y)}_{\mathbb{R}} = g(y) \cdot 0 = 0$$

so $gf \in I_Y$. //

<u>Upshot</u>: Every subset $Y \subset X$ gives rise to an ideal $I_Y \subset R$.

<u>Rmk</u> The fact that this example should become a paradigm is not obvious. For example, how do you think of $\mathbb{Z}$ as "functions on some space X"? How is the ideal $p\mathbb{Z}$ given by some "subset" of X?

Regardless, this kind of thinking has had huge influence on differential geometry, number theory (imagine being able to talk about the <u>geometry</u> of prime numbers!), et cetera.

Moreover, if we have $Y \subset X$, we should be able to talk about functions on Y — another ring!

<u>Philosophy</u> Let Y give rise to the ideal I_Y. Then

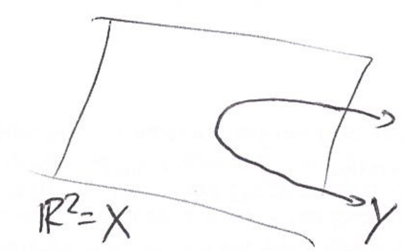
$$\{\text{fxns on } Y\} \cong R/I_Y.$$

<u>Example</u> (We deviate from all <u>continuous</u> functions, and just examine <u>polynomial</u> functions.)

Let $R = \mathbb{R}[x,y] = \{\text{polyn fxns on } \mathbb{R}^2\}$.

Let $Y = \{(x,y) \text{ s.t. } y^2 - x = 0\}$.

$=$



Then $I_Y = \{\text{polynomials } f \text{ such that } f(y^2, y) = 0\}$.

not obvious

$= (y^2 - x)$

↑ the ideal generated by the element $y^2 - x \in R$.

roughly: If ~~g~~ $f(x,y)$ vanishes on Y, it must be factored by $y^2 - x$.

Then $\{\text{algebraic/polynomial fxns on } Y\} \cong R/(y^2-x)$.

So why? If f_1, f_2 are functions on X, they restrict to functions on Y. But

$$f_1(y) = f_2(y) \quad \forall\, y \in Y$$

$$\Updownarrow$$

$$f_1(y) - f_2(y) = 0 \quad \forall\, y \in Y$$

$$\Updownarrow$$

$$f_1 - f_2 \in I_Y$$

i.e., f_1 and f_2 define the same function on Y iff $[f_1] = [f_2] \in R/I_Y$.

Polynomial rings :

Let F be a field.

Let $F[t]$ be the ring of polynomials.

Thm If $I \subset F[t]$ is an ideal, $\exists$ $p(t) \in F[t]$ such that

$$I = (p(t)).$$

I.e., every ideal is generated by a single element.

__Pf__ If $I = (0)$, done. So we can assume $\exists$ elements of degree ≥ 0.
Let $p(t)$ be an element of I with __least__ degree,

$$p(t) = a_0 + a_1 t + \cdots + a_d t^d, \quad d > -\infty.$$

Since $p(t) \in I$,

$$(p(t)) \subset I.$$

Now let $f(t) \in I$. Consider the division algorithm:

$$\begin{array}{r l}
 & \frac{b_n}{a_d} t^{n-d} + \frac{Q_{n-1}}{a_d} t^{n-d-1} + \frac{Q_{n-2}}{a_d} t^{n-d-2} + \cdots + \frac{Q_{n-d}}{a_d} t^0 \\
a_d t^d + \cdots + a_1 t + a_0 \, \big) & b_n t^n + b_{n-1} t^{n-1} + \cdots + b_{n-d} t^{n-d} + \cdots + b_1 t + b_0 \\
 & - \quad b_n t^n + \frac{a_{n-1} b_n}{a_d} t^{n-1} + \cdots \frac{a_0 b_n}{a_d} t^{n-d} + 0 + \cdots + 0 \\
\hline
 & (Q_{n-1}) t^{n-1} + \cdots + (\quad) t^{n-d} \cdots + (\quad) t + (\\
 & - \quad Q_{n-1} t^{n-1} + \frac{a_{n-1} Q_{n-1}}{a_d} t^{n-2} + \cdots + (\quad) t^{n-d-1} + 0 + \cdots + 0 \\
\hline
 & (Q_{n-2}) t^{n-2}
\end{array}$$

$Q_{n-1} = b_{n-1} - (a_d)^{-1} a_{n-1} b_n$

where $f(t) = b_n t^n + \cdots + b_0$.

$$p \,\overline{)\, f}^{\,q} + r.$$

Then

$$f(t) = p(t) q(t) + r(t)$$

where $\deg(r(t)) < \deg(p(t))$. But $p(t)$ had least degree, so $r(t) = 0$. //

Defn $f(t) \in F[t]$ is called <u>irreducible</u>, or <u>prime</u>, if

$$f(t) = a(t)\, b(t)$$

then either $a(t)$ or $b(t)$ is a constant polynomial.

I.e., if no polynomial degree d, $0 < d < \deg f$, divides f.

Thm Any $f(t)$ admits a factorization into irreducible polynomials.