MATH 122 NOTES – EMILY RIEHL

KEVIN YANG

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Prelim Remarks

Before I start I must say that today's lecture was fairly categorical, just as a preface. Moreover, I must admit that I added my own little flavor, but still stayed with Emily's presentation.

1. "Having some fun"

Suppose we have groups K, G, H, X with maps of groups

$$\begin{array}{ccc} K & \xrightarrow{\phi} & G \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\psi} \\ H & \xrightarrow{\beta} & X. \end{array}$$

We say this diagram *commutes* if $\psi \circ \phi = \beta \circ \alpha : K \to X$.

Remark 1.1. Notation: 1 will denote the trivial group. Check: this group is well-defined up to isomorphism! (It is, but just to make sure we have a good notion).

Now, look at the diagram

$$\begin{array}{ccc} K & \xrightarrow{\phi} & G \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle\psi} \\ 1 & \xrightarrow{!} & X \end{array}$$

Now what does it mean for this diagram to commute? This holds if and only if $\psi \circ \phi$ yields the trivial map that kills everything (sends everything to the identity). Why? Because the morphisms in the left and bottom are the trivial map that kills everything in K and the map that embeds the identity, respectively. Note that we didn't have to explicitly label the left and bottom maps. The maps are uniquely defined (assuming they're group homomorphisms), which is denoted by the !. So $\psi \circ \phi$ is the constant map at $1_X \in X$.

Alternatively, we have $\text{Im}(\phi) \subset \ker(\psi)$. Remember, this is close to what we had for exactness.

Ex 1.2. $K = G = X = 1$.

Ex 1.3. $K = SL_n(\mathbb{R})$, $G = GL_n(\mathbb{R})$, $X = \mathbb{R}^\times$, ϕ is the inclusion, and $\psi = \det$.

Ex 1.4. Let $K \subset G$ be a normal subgroup, so ϕ is the embedding. Let $X = G/K$ (note this is a group since K is normal in G), and ψ is the canonical quotient projection.

Theorem 1.5. *(Universal Property of the Quotient Group).*[1]

Given a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\text{normal}} & G \\ \downarrow & & \downarrow{\scriptstyle\psi} \\ 1 & \longrightarrow & X \end{array}$$

[1]This is the universal example – it turns out the above situation can be isomorphed to this situation.

There exists a unique homomorphism $\widetilde{\psi} : G/K \to X$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\psi} & X \\ {\scriptstyle\pi}\big\downarrow & & \big\uparrow{\scriptstyle\widetilde{\psi}} \\ G/K & \xrightarrow{\mathrm{Id}} & G/K \end{array}$$

such that this diagram commutes.[2]

Alternatively, if $\psi : G \to X$ is such that $K \subset \ker(\psi)$ for a $K \subset G$ a normal subgroup, then there exists a unique map $\widetilde{\psi} : G/K \to X$ such that $\psi = \widetilde{\psi} \circ \pi$, if $\pi : G \to G/K$ is the canonical map onto the quotient. This is called the universal property of the quotient group.[3]

Remark 1.6. Before we do this, I want to remark that this claim may seem scary and big, but it's actually a very natural and pretty easy construction, so don't fret!

Proof. We want to define this map $\widetilde{\psi} : G/K \to X$. Define it by taking a coset $Kg \mapsto \psi(g)$, or alternatively, taking the preimage of any coset in G/K back to G via the canonical map π (don't worry if this latter part doesn't make sense). We need to show that this is well-defined though.[4]

Say that $Kg = Kg'$, or equivalently $g = k \cdot g'$. Does $\psi(g) = \psi(g')$? Well

$$\psi(g) = \psi(k \cdot g') = \psi(k) \cdot \psi(g') = \psi(g'),$$

so everything works! We now need to check that $\widetilde{\psi}$ is unique, and that $\psi = \widetilde{\psi} \circ \pi$. But we have

$$\psi(g) = \widetilde{\psi}(Kg) = \widetilde{\psi}(\pi(g)),$$

so we just need to show uniqueness. But let's take another such map $\gamma : G/K \to X$, so we have

$$\psi(g) = \gamma(\pi(g)) = \gamma(Kg),$$

so that γ and $\widetilde{\psi}$ must agree on all cosets K, so we're done (note $\widetilde{\psi}$ is a homomorphism since everything else is). □

Remark 1.7. Note that uniqueness follows from the constraint of the commutativity of the diagrams in the statement of the theorem (or else it would not be unique), but that's all we care about! Moreover, this commutativity constraint gives a hint as to how we have to define this map to be.

2. Generalizations

This idea of a universal property allows us to "generalize" the notion of a quotient group, whatever that means...

Let $G \overset{\chi}{\twoheadrightarrow} H$ be any surjective homomorphism (we will now denote this with the two-head arrow). Then H is a "quotient group" of G, i.e. a diagram

$$\begin{array}{ccc} K & \longrightarrow & G \\ \big\downarrow & & \big\downarrow{\scriptstyle\chi} \\ 1 & \longrightarrow & H \end{array}$$

with the analogous universal property (think about **Ex. 1.4.**). To answer this, what is K? Let's just make it the kernel of χ, so that:

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[2]Note that there are a number of ways to draw this – I'm not good at TeXing them fast enough to do it now in lecture though...though on another note, note that saying a diagram commutes says that ANY two paths from ANY vertex to ANY OTHER vertex must agree (as maps).

[3]It turns out that universal properties are entirely ubiquitous in many areas of mathematics.

[4]There really is no other reasonable or natural construction/definition to make...and this turns out to be a thing in category theory – follow your nose!

Theorem 2.1. *Given $G \overset{\chi}{\twoheadrightarrow} H$, and let $K = \ker(\chi)$. Then the diagram*

$$\begin{array}{ccc} K & \xrightarrow{i} & G \\ \downarrow & & \downarrow \chi \\ 1 & \longrightarrow & H \end{array}$$

such that $i : K \to G$ is inclusion, has the same universal property, i.e. given any map $\psi : G \to X$ whose kernel contains K, we have a unique map $\widetilde{\psi} : H \to X$ such that

$$\psi = \widetilde{\psi} \circ \chi.$$

Proof. How do we define $\widetilde{\psi} : H \to X$. Well since χ is surjective, for any $h \in H$, we have $h = \chi(g)$, though g may not be unique! Well, now define

$$\widetilde{\psi} := \psi(g).$$

We need to check this is a well-defined homomorphism.[5] So let's say $\chi(g) = \chi(g')$. Then $\chi(g) = \chi(g')$ implies that $\chi(g \cdot (g')^{-1}) = 1_H$ so that $g \cdot (g')^{-1} \in K$, so that $g = g' \cdot k$ for some $k \in K$. The same arguments now just follow from the proof of the previous theorem. □

Corollary 2.2. *Suppose $\chi : G \to H$ is surjective and $K = \ker(\chi)$. Then $H \simeq G/K$.*

HA – First Isomorphism Theorem. But let's give another proof with universal properties.

Proof. **Theorem 1.5** gives a map $G/K \to H$, and **Theorem 2.1** gives a map $H \to G/K$. Composing these maps gives us a map $G/K \to H \to G/K$, such that the proper commutativity constraints are met. This map is given by composition of the two maps we get from the two theorems, but the identity map $G/K \to H$ also satisfies that constraint. Since the maps were uniquely defined, we must have that the composition IS the identity. Symmetrically, we obtain another map $H \to G/K \to H$ that is given by composing in the other direction, but again, $H \to H$ given by the identity also gives another map that meets the sufficient commutativity conditions, so the composition of those maps in the other order is the identity on H. So we have two maps $G/K \to H$ and $H \to G/K$ whose compositions in both orders are the respective identity maps, so that G/K and H are in bijection, but these maps were group homomorphisms, so they're actually isomorphic as groups![6] □

Remark 2.3. This notion of the universal property allows us to distinguish certain objects up to isomorphism uniquely, as we did in the proof of **Corollary 2.2**, since the universal property is a statement of existence AND uniqueness. If you're wondering, two objects "in a category" (whatever that means) are isomorphic if and only if they have the same universal property. In our example, two groups (the quotient group and any group surjected on by G) satisfied the same universal property in this "category of groups", so they were isomorphic![7]

2.1. **Challenge!** Let K, G be any groups, and let $\phi : K \to G$ be any group homomorphism. Then there exists another group homomorphism $\psi : G \to H$ such that we have a universal property

$$\begin{array}{ccc} K & \xrightarrow{\phi} & G \\ \downarrow & & \downarrow \psi \\ 1 & \longrightarrow & H. \end{array}$$

To answer this, we need to know H and ψ. Think about it, and if you want to check, see Emily!

Harvard College – Lowell House, Cambridge, MA 02138
E-mail address: kevinyang@college.harvard.edu

[5] Everything from here to the end of this proof was NOT said in class, but gives a hint as to how to complete the proof.
[6] Emily drew a bunch of diagrams that I can't TeX up in a reasonable amount of time.
[7] If you're more curious, this is a consequence of something very important in category theory called the *Yoneda Lemma*.