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Defn Let $K \subset G$ be a subgroup.

Then

$$N(K) = \{ g \in G \mid gKg^{-1} = K \}$$

is called the <u>normalizer</u> of K.

Prop'n

(1) $N(K)$ is a subgroup of G.

(2) $K \trianglelefteq N(K)$

(3) $N(K)$ is stabilizer of K with respect to the conjugation action of G on $\mathcal{P}_{|K|}(G)$.

Proof of First Sylow Thm:

G acts on $P_{p^e}(G)$ by left mult.

$$S \longmapsto gS.$$

Since $p \nmid |P_{p^e}(G)|$ by Lemma, $\exists$ orbit O_U s.t. $p \nmid O_U$. (Since $|P_{p^e}(G)| = \sum_{\text{orbits}} |O_U|$.)

Let $U \in O_U$. By Orbit-Stabilizer,

$$|G| / |G_U| = |O_U|$$

while by other lemma, $U = \bigcup_{\text{cosets}} xG_U$ so $|G_U|$ divides $|U| = p^e$.

Hence

$$p^e m = |G| = \underbrace{|G_U|}_{\text{power of } p} \cdot \underbrace{|O_U|}_{\text{not div. by } p}$$

$\Longrightarrow |G_U| = p^e$. //

Fix $H \in Syl_p(G)$. Suffices to show that for any $K < G$ w/ $|K| = p^\ell$, then $\exists\, g$ s.t. $K \subset gHg^{-1}$.

Proof of Second Sylow Thm:

Any subgroup $K \subset G$ acts on $G/H = \{\text{left cosets } gH\}$.

$$gH \longmapsto (kg)H.$$

If K is a p-group, and H is a Sylow p-subgroup,

$|K| = p^\ell$ while $|G/H| = |G|/|H| = p^e m / p^e = m$.

Since $p \nmid m$, the action of K on G/H has a fixed point: $\exists\, g$ s.t.

$$k \cdot gH = gH \quad \forall\, k \in K.$$

i.e., $\forall\, k \in K$, $\forall\, h \in H$, $\exists\, h' \in H$ s.t.

$$k \cdot g \cdot h = g \cdot h'$$

$\cancel{\Rightarrow \; g^{-1}kg = h'h^{-1}} \Rightarrow k \in gh'h^{-1}g^{-1}$

$\cancel{\Rightarrow \; g^{-1}kg \in H} \Rightarrow k \in gHg^{-1}$

$\Rightarrow K \subset gHg^{-1}$. //

Proof of Third Sylow Thm

Let H be a Sylow p-subgroup. ~~H~~ G acts on $Syl_p(G)$ by conjugation:

$$K \longmapsto hKh^{-1}.$$

By orbit-stabilizer,

$$|G| / |G_H| = |\mathcal{O}_H| = |Syl_p(G)|$$

$\mathcal{O}_H = Syl_p(G)$ by 2^{nd} Sylow Thm.

Well, $hHh^{-1} = H$ so $H \subset G_H$, so $p^e \,/\, |G_H|$.

Hence

$$p^e m / p^e m' = |Syl_p(G)|. \longrightarrow m = |Syl_p(G)| \cdot m'$$

Since G acts on $Syl_p(G)$, so does $H < G$. H is a fixed point. If K is another,

$$H \subset N(K) \text{ and } K \subset N(K).$$

If H is only fixed point,

$$|Syl_p(G)| = \underset{|\mathcal{O}_H|}{1} + \underset{\text{non-trivial orbits}}{\sum |\mathcal{O}|}$$

$$= 1 + Np, \text{ since } |\mathcal{O}| \text{ must divide } |H|.$$

By 2^{nd} Sylow, H is conjugate to K in $N(K)$. But $K \trianglelefteq N(K)$, so $H = K$. //