Friday, Oct 17, 2014

I know we were in the middle of semi-direct products, but it feels appropriate to have some more fun. So let's leave semidirect products for just a bit. (We'll see them again.)

<u>Recall</u>: Given $H \subset G$ a subgroup, we let
$$Hg = \{hg \mid h \in H\}$$
be the <u>right</u> coset of g.

We saw
$$Hg = Hg' \text{ iff } hg = g'$$
for some $h \in H$.

<u>Defn</u> We let
$$gH = \{gh \mid h \in H\}$$
be the <u>left</u> coset of g (with respect to H).

We'll also write G/H for this set; we will try, when possible, to never speak of right cosets again.

<u>Ex</u> Let $H = \langle (12) \rangle \subset S_3 = G$.

Then for $g = (123)$,
$$gH = \{(123), (123)(12)\}$$
$$= \{(123), (13)\}$$
while
$$Hg = \{(123), (12)(123)\}$$
$$= \{(123), (23)\}.$$

So $gH \neq Hg$ in general.

Prop'n

(1) $gH = g'H$ iff
$\exists\, h \in H$ s.t.
$gh = g'$

gH is the orbit for a group action from the right: $X \times H \to X$.

(2) $gH = Hg \;\forall\, g \in G$
iff
$H \triangleleft G$.

so our definition of the group G/H is the same — same elements, same operation.

Pf (1) $gH = g'H \Rightarrow \exists\, h_1, h_2 \in H$
s.t. $gh_1 = g'h_2$

$\Rightarrow gh_1h_2^{-1} = g'$.

Set $h_1h_2^{-1} = h$.

(2) $gH = Hg \Rightarrow \forall\, h \in H,$
$\exists\, h' \in H$ s.t.
$gh = h'g$

$\Rightarrow ghg^{-1} = h'$

ie, $\forall\, h \in H,\; ghg^{-1} \in H$

$\Rightarrow gHg^{-1} \subset H \;\forall\, g \in G$

$\Rightarrow H \triangleleft G$.

$H \triangleleft G \Rightarrow ghg^{-1} \in H \;\forall g \in G,\, h \in H$

$\Rightarrow \forall\, g \in G,\, h \in H,\, \exists\, h' \in H$ s.t.
$ghg^{-1} = h'$

$\Rightarrow \forall\, g \in G,\, h \in H,\, \exists\, h' \in H$
$gh = h'g$. //

We'll be discussing group actions — for fun!

Defn Let $\phi: G \to \mathrm{Aut}_{set}(X)$ be a group action.
Given $x \in X$, the *stabilizer* of x is the subgroup

$$G_x = \{g \mid gx = x\} = \{g \mid \phi_g(x) = x\} \subset G.$$

This is a subgroup since $\phi_g(x) = x, \phi_{g'}(x) = x$

$$\Rightarrow \underbrace{\phi_g(\phi_{g'}(x))}_{= \phi_g(x) = x} = \phi_{gg'}(x), \text{ and } \phi_{1_G}(x) = \mathrm{id}_X x = x.$$

Now, given an action of G on X, we have *two* things we can associate to an element $x \in X$:

$$\underset{\text{stabilizer}}{G_x \subset G}, \quad \underset{\text{orbit.}}{\mathcal{O}_x \subset X}$$

Prop'n The function

$$G/G_x \longrightarrow \mathcal{O}_x$$
$$gG_x \longmapsto gx = \phi_g(x)$$

is a bijection.

Cor If $|G_x|, |\mathcal{O}_x|$ are finite, so is $|G|$. Moreover, $|G| = |G_x||\mathcal{O}_x|$.

Pf Well-defined?

$$gG_x = g'G_x \Rightarrow g' = gh \quad \text{for some } h \in G_x$$
$$\Rightarrow g'x = (gh)x$$
$$= g(hx)$$
$$= gx$$

Surjective: $x' \in O_x \Rightarrow x' = gx$ for some $g \in G$.

Injective: $g'x = gx \Rightarrow g^{-1}g'x = x$
$$\Rightarrow g^{-1} \cdot g' \in G_x$$
$$\Rightarrow g' = gh \text{ for some } h \in G_x$$
$$\Rightarrow gG_x = g'G_x. \quad //$$

This is called the orbit-stabilizer theorem, and it's fantastic.

Example Let $P_n \subset \mathbb{R}^2$ be a regular n-gon in $\mathbb{R}^2$ centered at the origin.
Let $D_{2n} \subset GL_2(\mathbb{R})$ be the group of linear transformations s.t.

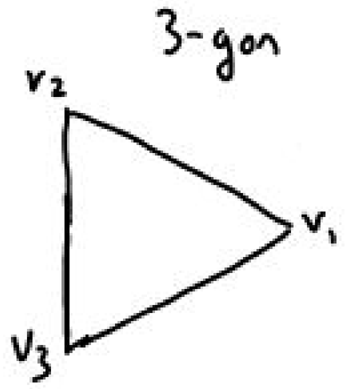
$$\forall\, g \in D_{2n}, \quad g(P_n) = P_n.$$

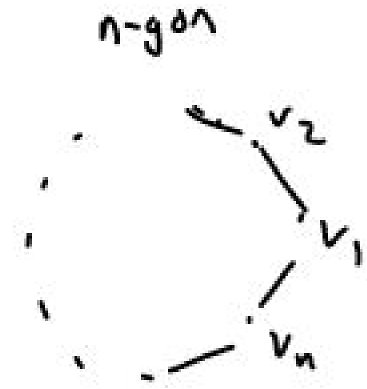
This means $g(P_n) \subset P_n$, $P_n \subset g(P_n)$.
Does NOT mean $g(x) = x \ \forall x \in P_n$.

ie, D_{2n} is the set of linear symmetries of P_n.

Claim $|D_{2n}| = 2n$.

Defn D_{2n} is called the n^{th} dihedral group.





The set P_n won't help — it has infinitely many elements. But if D_{2n} acts on P_n, it must permute the vertices $v_1, \cdots, v_n$ of P_n. So D_{2n} acts on the set $V = \{v_1, \cdots, v_n\}$.

Given, say, v_i, we can see that

$$\mathcal{O}_{v_i} = V.$$

Why? The rotation by $\frac{2\pi}{n}$ is linear, and sends P_n to P_n (since we chose P_n to be centered at the origin). Hence rotations by $\frac{2\pi k}{n}$ are in D_{2n}, and rotating v_i by $\frac{2\pi k}{n}$ hits every v_j.

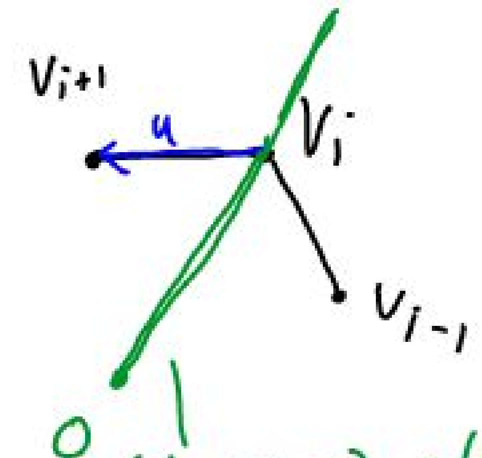
What's the stabilizer? If v_i is fixed, what could $g \in D_{2n}$ do to v_{i-1} and v_{i+1}?

- If $g(v_{i-1}) = v_{i-1}$, we have two lin. ind. vectors — v_i, v_{i-1} — fixed by g. Hence $g = \mathrm{id}_{\mathbb{R}^2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Otherwise,

- $g(v_{i-1}) = v_{i+1}$. Then g must be reflection about the line through $\vec{O}$ and v_i.

  the origin



$g(v_i) = v_i \Rightarrow g(tv_i) = tv_i \;\; \forall t \in \mathbb{R}$

$\Rightarrow$ this line is fixe

OTOH, $v_{i+1} = v_i + u$,

so $v_{i-1} = v_i - u$.

So g sends $u \mapsto -u$.

So $\exists$ exactly two elements of D_{2n} fixing v_i. i.e., the stabilizer of v_i has order two (hence is $\cong$ to $\mathbb{Z}/2\mathbb{Z}$).

By orbit-stabilizer theorem,

$$
\begin{aligned}
|D_{2n}| &= 2 \cdot |\mathcal{O}_{v_i}| \\
&= 2 \cdot |V| \\
&= 2 \cdot n .
\end{aligned}
$$

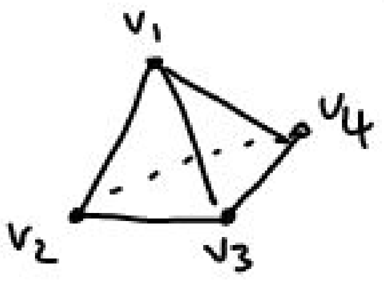
In your homework, you'll show that

$$D_{2n} \cong \underset{\text{rotations}}{\mathbb{Z}/n\mathbb{Z}} \rtimes \underset{\text{some reflection}}{\mathbb{Z}/2\mathbb{Z}}$$

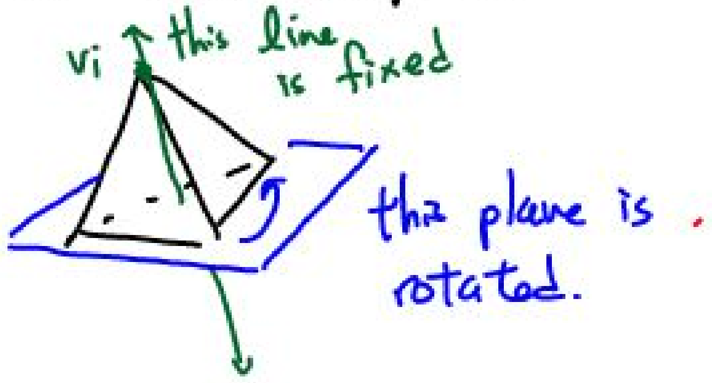
<u>Example</u> (Rotational symmetries)

Let T be a regular tetrahedon centered @ the origin. Let $G \subset SO_3(\mathbb{R})$ be the group of <u>rotations</u> g s.t. $g(T) = T$.

Well, G then acts on the set of <u>vertices</u> of T. T has four vertices

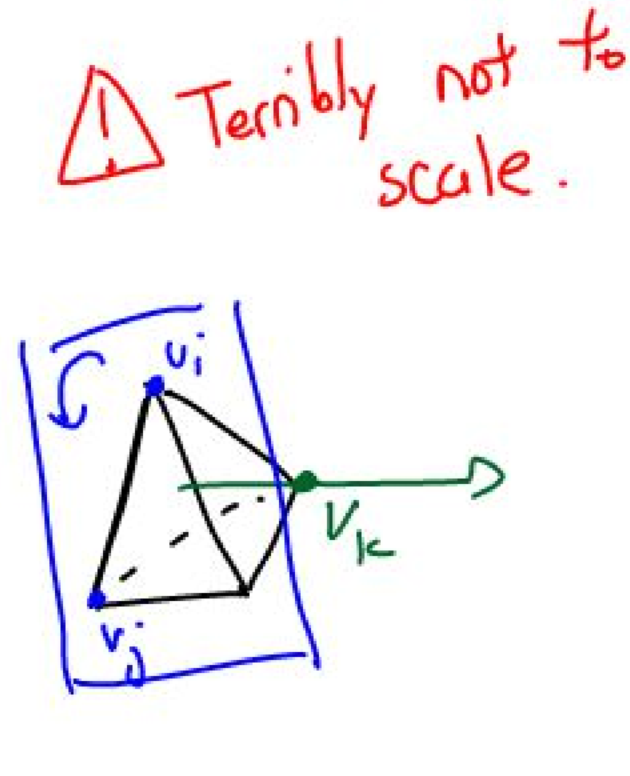


Let's first count the stabilizer of some v_i. Well, if g is a rotation fixing v_i, it must rotate the face opposite v_i, and fix the line through v_i.



Then there are three possible rotations of the plane — each by $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, or 0 radians. So the stabilizer of v_i is a group of order 3 (hence $\cong$ to $\mathbb{Z}/3\mathbb{Z}$).

What's the orbit? Every vertex! For if you want to find g s.t. $g(v_i) = g(v_j)$, choose g to be a rotation fixing some v_k, $v_k \neq v_i, v_j$, and rotate!



By orbit-stabilizer theorem,

$$\begin{aligned} |G| &= 3 \cdot |\mathcal{O}_{v_i}| \\ &= 3 \cdot 4 \\ &= 12. \end{aligned}$$

We'll see, eventually, what group this is.