CHAPTER 15

Simple groups and short exact sequences

Chit-chat 15.1. Now that we've seen examples of many groups, we'd like to start classifying them. Can we think of a general strategy that will help us say: "I know all groups?"

This question doesn't have a satisfactory answer, in some ways. The strategy that took flight in the 19th century is called the *Hölder program*, which I'll describe in a little bit.

1. Extensions, a.k.a. short exact sequences

Definition 15.2. A *short exact sequence* of groups is a pair of homomorphisms

$$G \to H \to K$$

such that

(1) $G \to H$ is an injection,
(2) $H \to K$ is a surjection, and
(3) the kernel of $H \to K$ is *equal* (not just isomorphic) to the image of $G \to H$.

A short exact sequence is often written

$$1 \to G \to H \to K \to 1.$$

Definition 15.3. We will also say that H *is an extension of* K *by* G.

Chit-chat 15.4. The reason for the 1 on the ends? The 1 represents the trivial group with one element. The above sequence is "exact" in the sense that the image of any homomorphism is the kernel of the next. For instance, the portion $1 \to G \to H$ says that the image of $1 \to G$ is the kernel of $G \to H$—i.e., $G \to H$ is injective.

Chit-chat 15.5. For reasons that will become clearer later, short exact sequences are important because of the following philosophy: *We think of the group H as built up of the groups G and K.*

Example 15.6. There are short exact sequences

(1) $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, and [1]
(2) $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. [2]

So we can think of both $\mathbb{Z}/4\mathbb{Z}$ and the Klein four group as built out of two copies of $\mathbb{Z}/2\mathbb{Z}$, but we see there are different groups we can build out of $\mathbb{Z}/2\mathbb{Z}$.

We also have short exact sequences

(3) $\mathbb{Z}/3\mathbb{Z} \to S_3 \to \mathbb{Z}/2\mathbb{Z}$,[3] and
(4) $\mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.

So we see that there are at least two different ways to build a group of order 6 out of $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$.

Remark 15.7. The above examples show:

(1) Extensions do not need to be direct products.
(2) An extension $G \to H \to K$ may not allow for maps $H \leftarrow K$ such that $K \to H \to K = id_K$.
(3) Extensions of abelian groups can be non-abelian.

2. Simple groups

Chit-chat 15.8. But there are some groups that can't be built out of any others. For instance, what if H doesn't allow for any (non-trivial) normal subgroups? Then it's impossible to have a short exact sequence unless $H \cong K$ or $G \cong H$. In this sense, groups without normal subgroups are the simplest groups.

Definition 15.9. A group H is called *simple* if it has no non-trivial normal subgroups.

Example 15.10. A cyclic group is simple if and only if it has finite, prime order. [4]

Example 15.11. $\mathbb{Z}$ is not simple. [5]

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[1] The first homomorphism sends $1 \mapsto 2 \in \mathbb{Z}/4\mathbb{Z}$, while the second sends $0, 2 \mapsto 0$ and $1, 3 \mapsto 0 \in \mathbb{Z}/2\mathbb{Z}$.

[2] We send $a \mapsto (a, 0)$ and $(a, b) \mapsto b$.

[3] $A_3 \cong \mathbb{Z}/3\mathbb{Z}$, and this is simply the short exact sequence associated to the inclusion $A_3 \to S_3$.

[4] If it has prime order, it is simple since it has no subgroups but itself and $\{1\}$. On the other hand, a cyclic group of order n has a subgroup for every number n/k dividing n; for instance, take $\{1, x^k, \ldots\}$ for any generator x. Hence a cyclic group is simple when it has prime order.

[5] It has many subgroups, and any subgroup of an abelian group is normal.

Example 15.12. A_1, A_2, A_3 are simple. [6]

Example 15.13. A_4 is *not* simple. [7]

The following is one of the big theorems we'll prove in this class:

Theorem 15.14. A_n is simple for all $n \geq 5$. [8]

3. The Hölder program

Chit-chat 15.15. So how can we classify all groups? We can try to understand all *simple* groups, and then understand ways in which they can all be built up. This is called the *Hölder program*. It seems very natural. The problem is, we don't know how to execute it. We can't even classify all infinite simple groups.

Can we understand, then, all *finite* simple groups and their extensions? This will at least classify all finite groups. We still don't know how to do this. We can *classify* all finite simple groups as of 1985-ish, but we still don't know how to solve the problem of classifying all their extensions. To give you some idea of how difficult even classification is, consider that the following theorem led to a Fields Medal for Thompson:

Theorem 15.16 (Feit-Thompson, or the Odd Order Theorem)**.** Every finite, simple, non-abelian group has even order.

Chit-chat 15.17. So for instance, if you give me a non-abelian group whose order is odd, I know it's not simple.

Definition 15.18. The *Hölder program* for classifying groups is:

(1) Classify all simple groups.
(2) Classify all the ways that one can create extensions of simple groups.

Chit-chat 15.19. We don't know how to complete this program. For instance, we don't know how to do (1). We only know how to do (1) for *finite* simple groups, and this wasn't done until 1985. Even for finite groups, we haven't done (2).

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[6] A_1 and A_2 both have one element. A_3 is a subgroup of S_3 or index 2 by first isomorphism theorem—it is a group of order 3, which is cyclic of prime order.

[7] We need to exhibit a non-trivial normal subgroup. There is a unique one: It consists of elements that are products of 2-cycles. This normal subgroup is isomorphic to the Klein four group, and the quotient group is the cyclic group of order 3.

[8] (Proof will come eventually.)

4. Split short exact sequences

Definition 15.20. We say a short exact sequence

$$G \to H \to K$$

splits if there is a homomorphism $K \to H$ such that $K \to H \to K = id_K$. Of the examples of the short exact sequences from above

(1) $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$,
(2) $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.
(3) $\mathbb{Z}/3\mathbb{Z} \to S_3 \to \mathbb{Z}/2\mathbb{Z}$,
(4) $\mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.

only (1) does *not* split.

Chit-chat 15.21. Next time, we'll talk more about split exact sequences.