The Fundamental Group

This lecture is for fun. Many statements are made at a non-rigorous level. Just enjoy the ride!

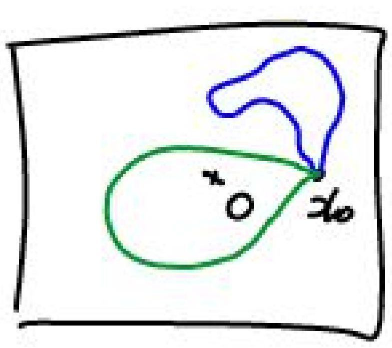
Defn Let $X \subset \mathbb{R}^n$ be a subset, and fix $x_0 \in X$. A loop in X based at x_0 is a continuous function

$$[0,1] \xrightarrow{\gamma} \mathbb{R}^n$$

such that

- $\gamma(t) \in X \ \forall \ t \in [0,1]$
- $\gamma(0) = \gamma(1) = x_0$.

Ex $X = \mathbb{R}^2 \setminus \{0\}$, $x_0 = (1,0)$.



X

Defn Given two curves γ_1 and γ_2, we say γ_1 and γ_2 are <u>homotopic</u> if γ_1 can be wiggled into γ_2 w/out changing $\gamma(0)$ and $\gamma(1)$.

Defn i.e., if $\exists$

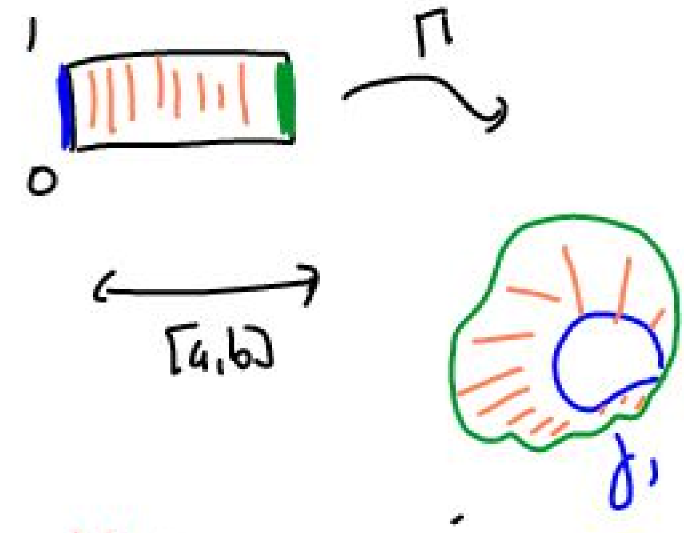
Continuous map (some interval)

$$\Gamma: [0,1] \times [a,b] \longrightarrow X$$
$$(t, s) \longrightarrow \Gamma(t,s)$$

such that

- $\Gamma(t, a) = \gamma_1(t)$
- $\Gamma(t, b) = \gamma_2(t)$
- $\Gamma(0, s) = \Gamma(1, s) = x_0$ $\forall s$.

You can think of Γ as some "movie" of paths lasting $b-a$ seconds.



Lemma Say $\gamma_1 \sim \gamma_2$ iff γ_1 is homotopic to γ_2. This is an equivalence relation.

Roughly:

- Any path can be wiggled to itself by the boring wiggle. (No wiggling at all!)
- If γ_1 wiggles to γ_2, the reverse wiggle will wiggle γ_2 to γ_1.
- If γ_1 wiggles to γ_2, and γ_2 wiggles to γ_3, just perform the two wiggles to exhibit a single wiggle from γ_1 to γ_3.

Defn Let

$$\pi_1(X, x_0) = \{\text{loops at } x_0\} / \sim .$$

This is called the <u>fundamental</u> <u>group</u> of X with basepoint x_0.

Rmk If X is connected, then $\pi_1(X, x_0) \cong \pi_1(X, x_0')$ for any two basepoints.

Composition is as follows:

$$\pi_1(X,x_0)\times\pi_1(X,x_0)\longrightarrow\pi_1(X,x_0)$$
$$([\gamma_b],[\gamma_a])\longmapsto[\gamma_b][\gamma_a]$$

Given two paths γ_b and γ_a, consider the path

$$\widetilde{\gamma_b\circ\gamma_a}:[0,2]\longrightarrow X$$
$$t\longmapsto\begin{cases}\gamma_a(t) & \text{if } t\in[0,1]\\ \gamma_b(t-1) & \text{if } t\in[1,2].\end{cases}$$

Do γ_a, then do γ_b.

Rescale $[0,2]$ to $[0,1]$ to obtain a path

$$\gamma_b\circ\gamma_a:[0,1]\longrightarrow X$$
$$t\longmapsto\begin{cases}\gamma_a(2t) & t\in[0,\tfrac12]\\ \gamma_b(2t-1) & t\in[\tfrac12,1]\end{cases}$$

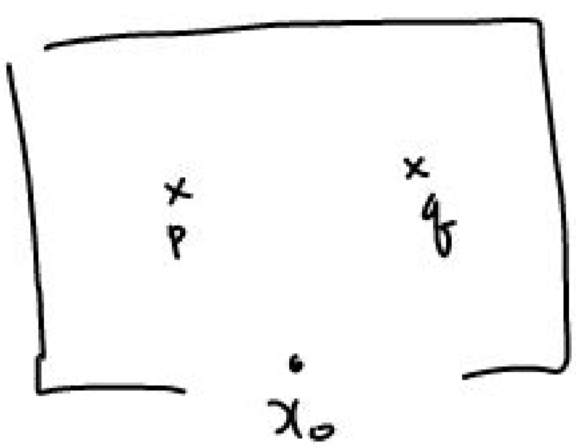
We let

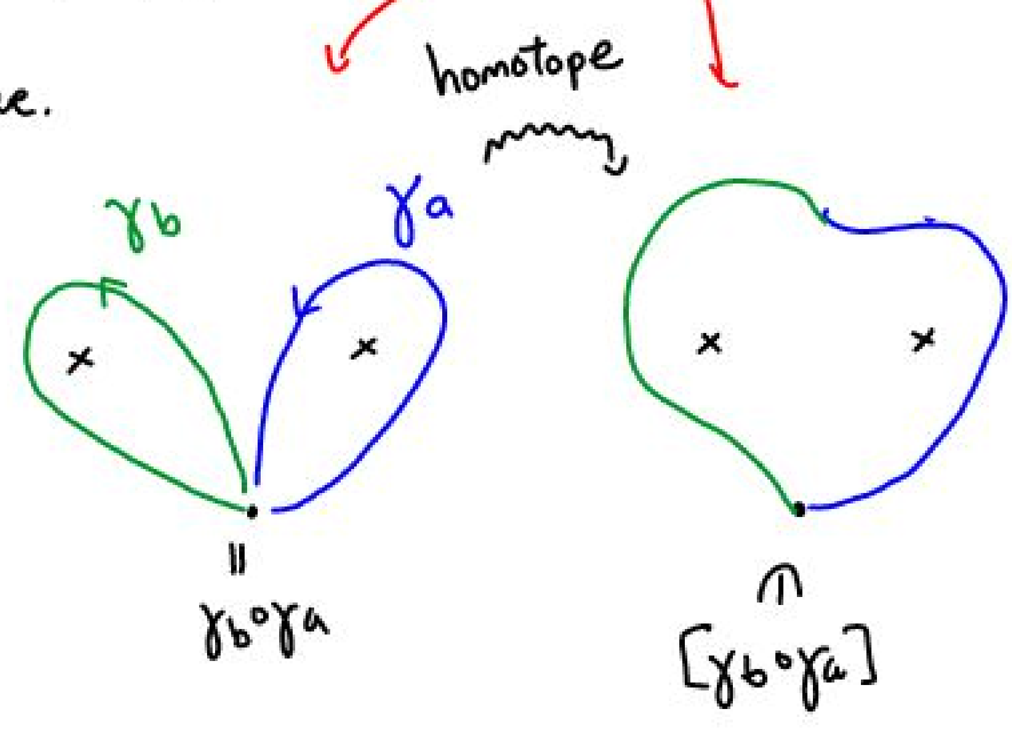
$$[\gamma_b][\gamma_a]=[\gamma_b\circ\gamma_a].$$

these are both representatives of $[\gamma_b\circ\gamma_a]$.

<u>Ex</u> Consider $X=\mathbb{R}^2\setminus\{p,q\}$

x_0 anywhere.



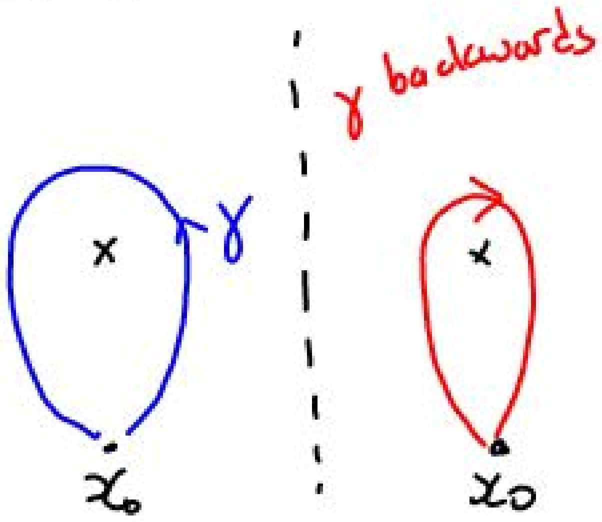


Rmk · The constant loop

$$\gamma(t) = x_0 \quad \forall t$$

is the identity.

· The inverse of γ is the loop called "do γ backwards."

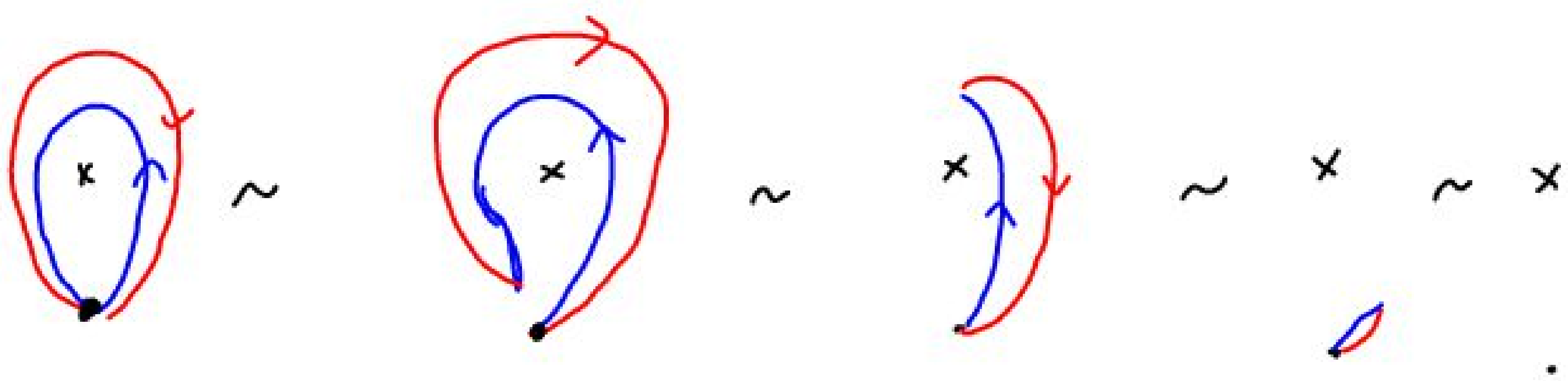


– If γ is a loop and γ_0 is the constant loop,

$$\gamma_0 \circ \gamma$$

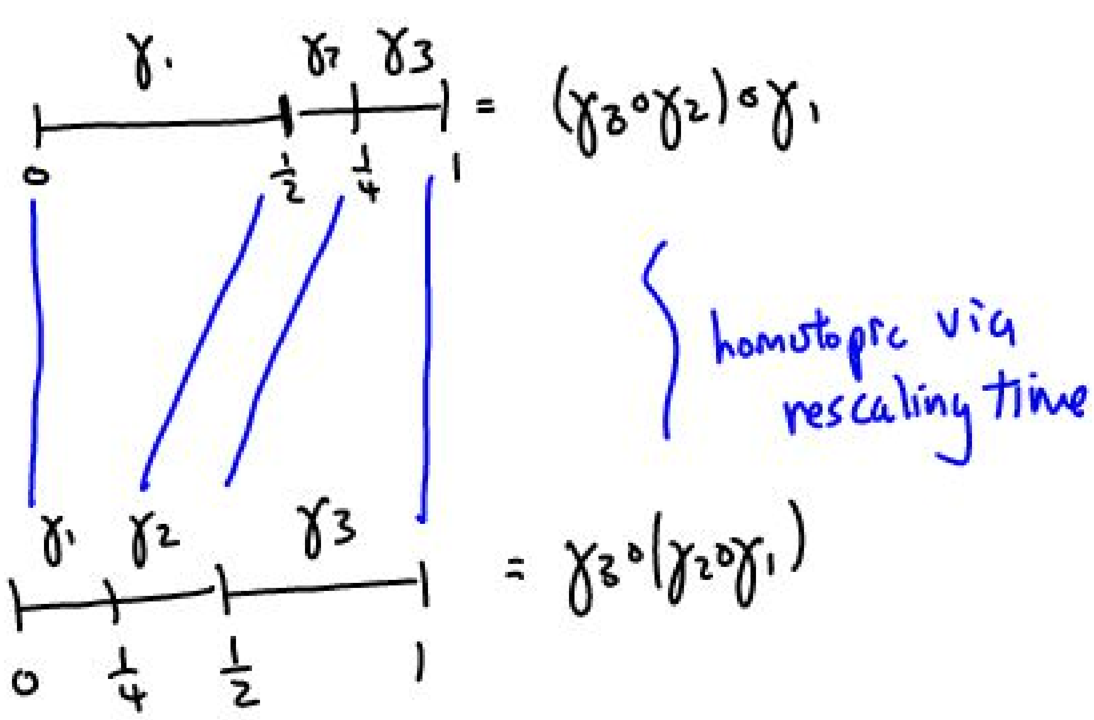
is the loop called "Do γ quickly, then do nothing for ½ a second." Well, let Γ be the homotopy that shrinks this "do nothing" time from ½ to 0 seconds.

$$\left[\text{"}\gamma \text{ backwards"} \circ \gamma\right] = ?$$



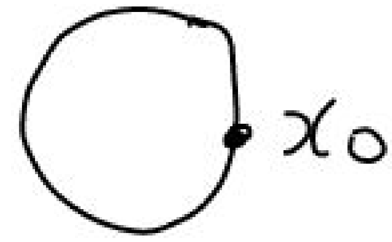
constant loop!

· Finally, composition is associative.



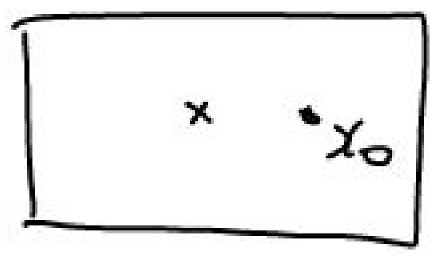
Ex

$\pi_1(S^1, x_0)$



$\cong$

$\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$



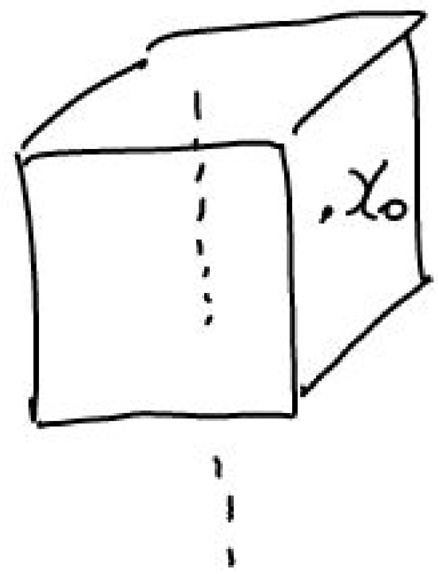
$\cong$

$\pi_1(\mathbb{R}^3 \setminus L, x_0)$.

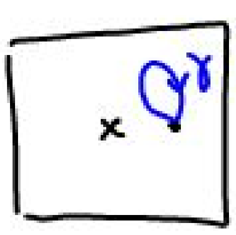
a line

$\cong$

$\mathbb{Z}$.

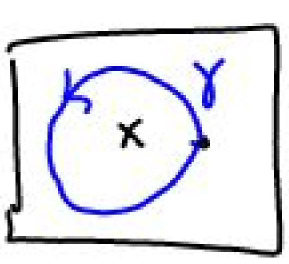


I'm not proving this isomorphism for you — but roughly, you send a path γ to its "winding number."



$\longmapsto 0 \in \mathbb{Z}$

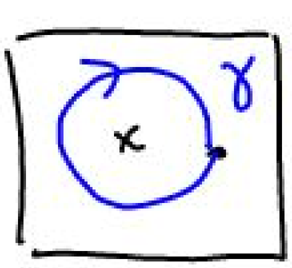
"how many times does γ wind around the puncture?"



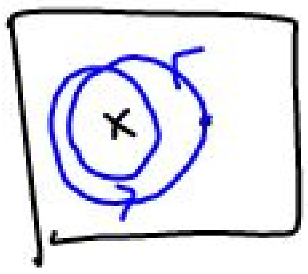
$\longmapsto 1$

Ex If γ is differentiable, you can define this as

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z}.$$



$\longmapsto -1$

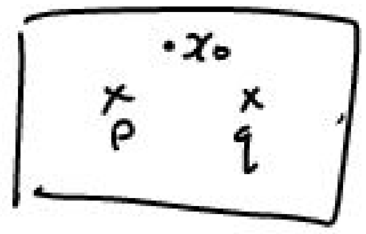


$\longmapsto 2$

Ex $\pi_1(\infty, x_0)$

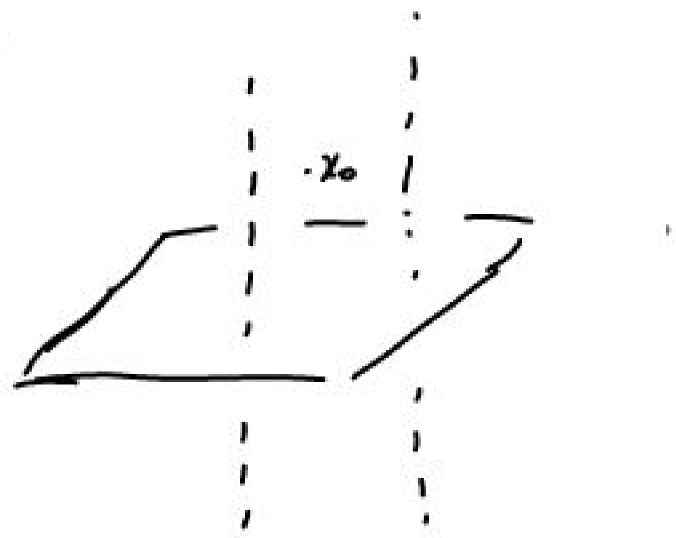
$\cong$

$\pi_1(\mathbb{R}^2 \setminus \{p,q\}, x_0)$



$\cong$

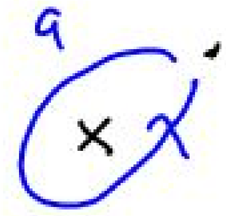
$\pi_1(\mathbb{R}^3 \setminus (L \cup L'), x_0)$

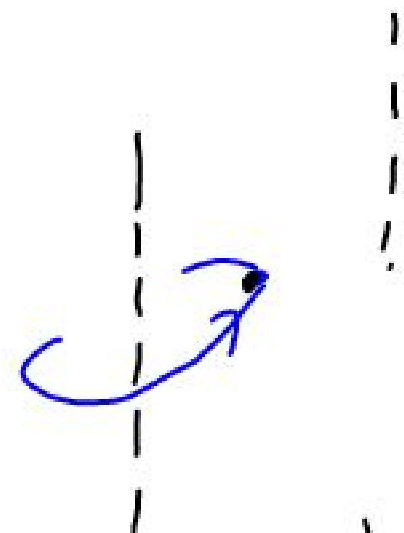


$\cong$

Free group on two generators.

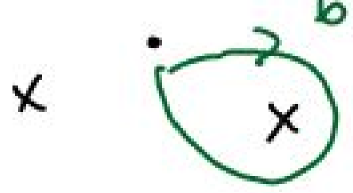
a

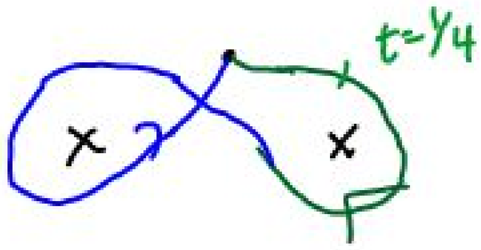




a^{-1}

b

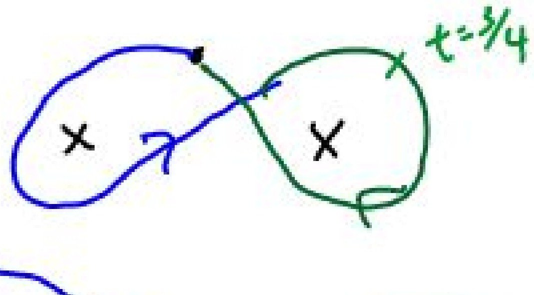


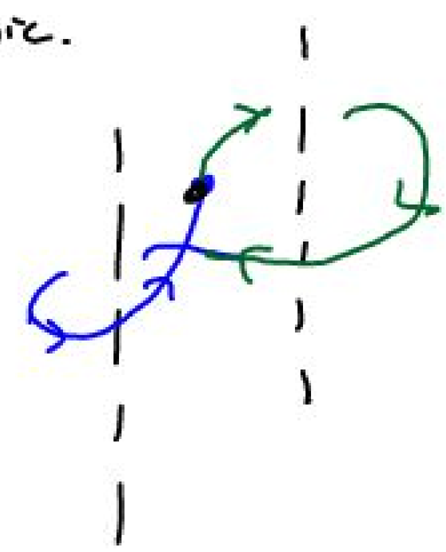


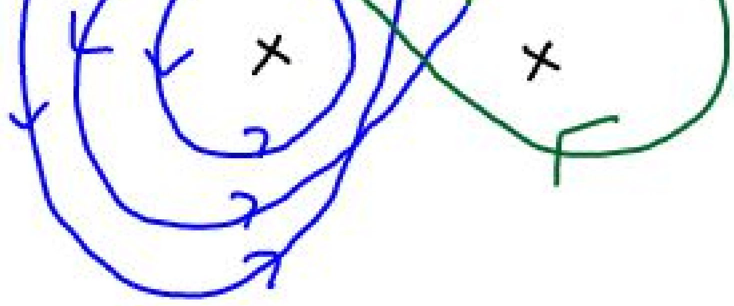
ab

NOT homotopic.

ba







$abaa$