Cycle notation gives us some nice conceqeunces

Propn (1) Let $\sigma \in S_n$ be a cycle, so

$$\sigma = (a_1 \cdots a_k)$$

where $a_{i+1} = \sigma(a_i)$. Then

$$\sigma^{-1} = (a_k \cdots a_1).$$

i.e., $\sigma^{-1} = (b_1 \cdots b_k)$ where $b_i = \sigma(b_{i+1})$, and $b_k = a_1$.

(2) More genrally, if

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_k \quad \text{where } \sigma_i$$

are disjoint cycles, then

$$\sigma^{-1} = \sigma_1^{-1} \cdots \sigma_k^{-1}.$$

(3) Let $\sigma, \tau \in S_n$ and $a, b \in \underline{n}$.
If $\sigma(a) = b$, then $\tau\sigma\tau^{-1}$ sends $\tau(a)$ to $\tau(b)$.

Rmk Conjugation is like a change of basis. If $v_1, \ldots, v_k$ are a basis for $\mathbb{R}^k$, one has an invertible matrix T whose i^{th} column is v_i. If a linear transformation A sends $\vec{a}$ to $\vec{b}$, then TAT^{-1} sends $T\vec{a}$ to $T\vec{b}$.
So think of τ above as giving a "new basis" to $\underline{n}$.

Pf (1) NTS that $\forall\ b \in \underline{n}$,

we have

$$(a_k \cdots a_1) \circ (a_1 \cdots a_k) : b \mapsto b$$

and

$$(a_1 \cdots a_k) \circ (a_k \cdots a_1) : b \mapsto b$$

We'll do the first composition; leave the second to you. Note:

- $b \notin \{a_1, \dots, a_k\}$
  $\Rightarrow b$ is fixed by σ
  $\Rightarrow b$ is fixed by $(a_k \cdots a_1)$
  $\Rightarrow (a_k \cdots a_1) \circ \sigma(b) = b.$ ✓
- $b \in \{a_1, \dots, a_k\}$
  $\Rightarrow b = a_i$ for some $i \in 1, \dots, k$
  $\Rightarrow \sigma(b) = a_{i+1}$ (defn of cycle notation)
  $\Rightarrow (a_k \cdots a_1)$ sends $\sigma(b)$ to b. ✓

(2) In general, if $g_1, \dots, g_\ell \in G$,

$$(g_1 \cdots g_\ell)^{-1} = g_\ell^{-1} \cdots g_1^{-1}.$$

Why?

$$\begin{aligned}(g_1 \cdots g_\ell)(g_\ell^{-1} \cdots g_1^{-1}) &= g_1 \cdots g_{\ell-1} \underbrace{g_\ell g_\ell^{-1}}_{\text{cancel}} g_{\ell-1}^{-1} \cdots g_1^{-1} \\ &= g_1 \cdots \underbrace{g_{\ell-1} g_{\ell-1}^{-1}}_{\text{cancel}} \cdots g_1^{-1} \\ &\ \vdots \\ &= g_1 g_1^{-1} \\ &= 1_G.\end{aligned}$$

So $(\sigma_1 \cdots \sigma_\ell)^{-1} = \sigma_\ell^{-1} \cdots \sigma_1^{-1}$.

But disjoint cycles commute, so

$$\sigma_\ell^{-1} \cdots \sigma_1^{-1} = \sigma_1^{-1} \cdots \sigma_\ell^{-1}.$$

(3)

$$\tau\sigma\tau^{-1}(\tau(a)) = \tau\sigma\tau^{-1}\circ\tau(a)$$
$$= \tau\sigma(a)$$
$$= \tau(b). //$$

<u>Cor</u> Let $\sigma, \sigma' \in S_n$.
If $\sigma' = \tau\sigma\tau^{-1}$
for some $\tau \in S_n$, then we can write the cycle notation for σ' from the cycle notation for σ, and from τ.

(3) tells us that $\tau(a_i)$ is sent to $\tau(a_{i+1})$ by $\tau\sigma\tau^{-1}$. It also tells us that $\sigma(a) = a \Rightarrow \tau\sigma\tau^{-1}$ fixes $\tau(a)$; so $\tau\sigma\tau^{-1}$ is another cycle whose non-trivial orbit is given by $\{\tau(a_i)\}$.

<u>Pf</u> If σ is a cycle,
$$\sigma = (a_1 \cdots a_\ell)$$
then
$$\tau\sigma\tau^{-1} = (\tau(a_1) \cdots \tau(a_\ell)).$$

If σ is a product
$$\sigma = \sigma_1 \cdots \sigma_\ell$$
of disjoint cycles,
$$\tau\sigma\tau^{-1} = (\tau\sigma_1\tau^{-1})(\tau\sigma_2\tau^{-1})\cdots(\tau\sigma_\ell\tau^{-1})$$ since conjugation by τ is a group homomorphism

so if
$$\sigma = (a_1 \cdots a_{k_1})(a_{k_1+1} \cdots a_{k_1+k_2}) \cdots (a_{k_1+\cdots+k_{l-1}+1} \cdots a_{k_1+\cdots+k_\ell})$$
is a cycle notation for σ, then
$$\tau\sigma\tau^{-1} = (\tau(a_1) \cdots \tau(a_{k_1}))(\tau(a_{k_1+1}) \cdots \tau(a_{k_1+k_2})) \cdots (\tau(a_{k_1+\cdots+k_{l-1}+1}) \cdots \tau(a_{k_1+\cdots+k_\ell}))$$
is a cycle notation for $\tau\sigma\tau^{-1}$.

Let $\sigma \in S_n$.

Write σ as a product

$$\sigma = \sigma_1 \cdots \sigma_k$$

of disjoint cycles, and consider

$$|\sigma_i| \quad \forall\, i.$$

(These are the sizes of the orbits associated to each σ_i.)

In this way, we get some collection of numbers. It's most conveniently thought of as an <u>unordered</u> collection, since we can reorder the σ_i.

<u>Ex</u> Let

$$\sigma = (123)(67)(459) \in S_9.$$

Note we don't write (8), for sake of brevity. Then we have numbers

$$3,\ 2,\ 3$$

associated to σ.

<u>Defn</u> We call the numbers $\{a_i\}$ the <u>cycle</u> <u>shape</u> of σ.

Ex Let $\sigma_1' = (345)(879)(26)$

Then σ_1' has numbers 3, 3, 2 associated to it. Up to reordering, this is the same collection as for σ. We say σ and σ' have the <u>same cycle shape</u>.

Prop Two elements $\sigma, \sigma' \in S_n$ are conjugate — ie, $\exists$ τ s.t. $\sigma = \tau\sigma'\tau^{-1}$! iff they have the same cycle shape.

Pf Let σ and σ' have the same cycle shape. We can then reorder any cycle notation for σ and σ' so

$$\sigma = \sigma_1 \circ \cdots \circ \sigma_k$$

$$\sigma' = \sigma_1' \circ \cdots \circ \sigma_k'$$

← product of disjoint cycles.

where $|\sigma_i| = |\sigma_i'|$ $\forall$ i.

Choose any i, and any number a that appears in the cycle notation for σ_i.

$$\sigma_i = (\cdots\cdots a \cdots\cdot).$$

In σ_i', choose any number a'.

$$\sigma_i' = (\cdots a' \cdots\cdots).$$

Define a bijection as follows:

$$\tau : a_i \longmapsto b_i$$
$$\sigma^j(a_i) \longmapsto (\sigma')^j(b_i).$$

Then

$$\tau\sigma\tau^{-1}(b) = \tau\sigma\tau^{-1}((\sigma')^j(b_i))$$
$$= \tau\sigma(\sigma^j(a_i))$$
$$= \tau(\sigma^{j+1}(a_i))$$
$$= (\sigma')^{j+1}(b_i)$$
$$= \sigma'(b).$$

ie, $\tau\sigma\tau^{-1} = \sigma'$.

The converse follows from the corollary.

Ex:

$$(123)(69) = \sigma$$
$$(45)(361) = \sigma' \quad \in S_9$$

have the same cycle shape.

As do

$$\sigma = (12)(34)(567)$$
$$\sigma' = (78)(59)(142) \in S_9.$$

Rmk The cycle shape of σ just says: The action of σ breaks $\underline{n}$ into l many orbits; the "i^{th}" orbit has size k_i. If σ' also breaks $\underline{n}$ into l many orbits, and we can match up their sizes k_i' to those k_i of σ, then σ and σ' have the same cycle shape.

Ex How might you find τ?

$$\sigma = (123)(46)(785)$$
$$\sigma' = (157)(93)(684).$$

Well, if $\tau\sigma\tau^{-1} = \sigma'$, we know that a cycle

$$(b_1 \cdots b_k)$$

in the cycle notation for σ' equals

$$(\tau(a_1) \cdots \tau(a_k))$$

for some cycle $(a_1 \cdots a_k)$ of σ's cycle decomposition. This τ is NOT unique, but here's how you can find it:

Pick a cycle, and a number appearing in a cycle notation for it. For no reason, let's choose

$$4 \in (46).$$

Choose a cycle in σ''s cycle notation, w/ same length as (46). In this case, we're constrained to (93) (though in general, we may have many choices). Choose an element appearing in this cycle, say 9.

$$(123)(46)(785)$$
$$\overset{②}{\swarrow} \quad \overset{①}{\downarrow}$$
$$(157)(93)(684).$$

So write

$$\tau = (4 \overset{①}{\curvearrowright} 9$$

then see what cycle σ_i contains 9. None in this case – 9 is a fixed pt of σ. So choose any fixed pt of σ' — here, our only choice is 2.

$$\tau = (4 \overset{①}{\curvearrowright} 9 \overset{②}{\curvearrowright} 2$$

Now find a cycle σ_i that contains 2. In σ_i', find corresponding element. In this case, it's 5.

$$\tau = (4 9 2 \overset{③}{\curvearrowright} 5$$

③ After seeing (4925) is a cycle for τ, just choose any element not yet written. We chose 1 arbitrarily.

So forth.

$$(123)(46)(785)$$
$$(157)(93)(684)$$

④ ⑤ ⑥ ⑦ ⑧

⑥ Likewise.

$$\tau = (4925)(1)(3 \overset{⑦}{\curvearrowright} 7 \overset{⑧}{\curvearrowright} 6)(8)$$
$$= (4925)(376).$$

since we never write cycles of length 1.

Defn The alternating group

$$A_n$$

is defined to be the kernel of the map

$$S_n \longrightarrow GL_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^\times.$$

$$\sigma \longmapsto B_\sigma \text{ s.t. } B_\sigma(e_i) = e_{\sigma(i)}$$

i.e, the collection of all σ s.t. B_σ has det 1.

Thm A_n is a simple group for $n \geq 5$

We'll define "simple" soon.