FRI, Sept 26, 2014

Defn Let

$$G \to \mathrm{Aut}_{Set}(X)$$

be a grp action of G on X. Fix $g \in G$. By <u>the action of g on X</u>, we mean the action

$$\langle g \rangle \to G \to \mathrm{Aut}_{Set}(X).$$

composition of two gp homom is a gp homom.

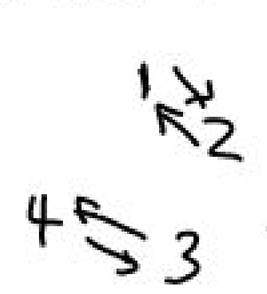
inclusion of a subgp is a gp homom

Defn An element $\sigma \in S_n$ is called a <u>cycle</u> if σ's action on $\underline{n}$ has at most one orbit of size ≥ 2.

Ex • $\sigma = 1_{S_n}$ has only orbits of size 1. So 1_{S_n} is a cycle.

• Let $\tau : \underline{4} \to \underline{4}$, which we'll draw as

$$\begin{aligned} 1 &\mapsto 2 \\ 2 &\mapsto 1 \\ 3 &\mapsto 4 \\ 4 &\mapsto 3 \end{aligned}$$

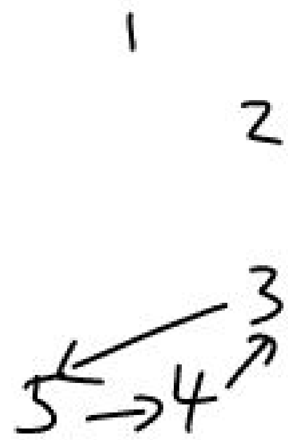


This is <u>NOT</u> a cycle, as it has <u>two</u> orbits of size ≥ 2: $\{1,2\}$ and $\{3,4\}$.

Ex

$\tau : \underline{5} \longrightarrow \underline{5}$

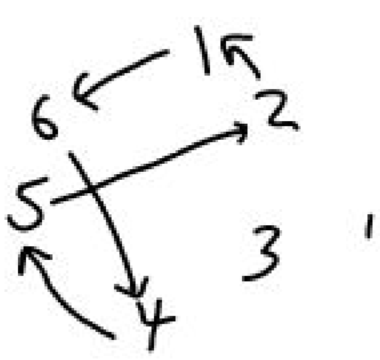
$1 \longmapsto 1$
$2 \longmapsto 2$
$3 \longmapsto 5$
$4 \longmapsto 3$
$5 \longmapsto 4$

,  , is a cycle.

Defn If $\sigma \in S_n$ is a cycle, we'll let $\underline{\sigma} \subset \underline{n}$ denote the orbit of size ≥ 2.

For $\sigma = 1_G$, we let
$$\underline{1_G} := \emptyset.$$

Ex For $\sigma \in S_6$ given by

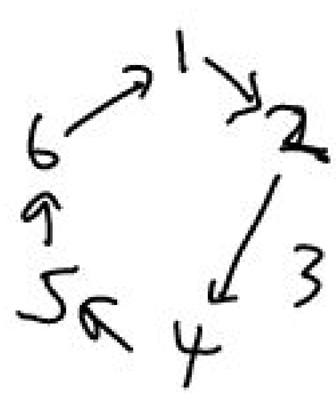


,

$\underline{\sigma} = \{1, 2, 5, 4, 6\} \subset \underline{6}$.

a subset, so order of elements don't matter. e.g., it's not some set together w/ a choice of ordering.

Ex If $\tau \in S_6$ is



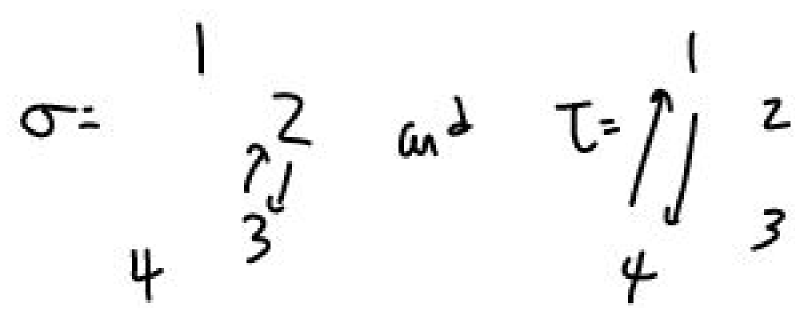
,

$\underline{\tau} = \underline{\sigma}$.

Def'n Suppose $\sigma, \tau \in S_n$ are cycles. We say σ and τ are <u>disjoint</u> <u>cycles</u> iff $\underline{\sigma}$ and $\underline{\tau}$ don't intersect.

Ex · 1_G is disjoint from any cycle.

are NOT disjoint, since $\{1,2\}$ and $\{2,3\}$ intersect.



are disjoint.

Prop'n Disjoint cycles in S_n commute.

Pf: Let σ, τ be disjoint cycles and let $k \in \underline{n} = \{1, \dots, n\}$.

$k \in \underline{\tau} \Rightarrow \tau(k) = k$

$\Rightarrow \sigma\tau(k) = \sigma(k)$.

Then

$$(\sigma \circ \tau)(k) = \begin{cases} \sigma(k) & \text{if } k \notin \underline{\tau} \\ \tau(k) & \text{if } k \in \underline{\tau} \end{cases}$$

$k \in \underline{\tau} \Rightarrow \tau(k) \in \underline{\tau}$ defn of orbit

$\Rightarrow \tau(k) \notin \underline{\sigma}$ defn of disjoint

$\Rightarrow \sigma(\tau(k)) = \tau(k)$

$$= \begin{cases} \sigma(k) & \text{if } k \in \underline{\sigma} \\ k & \text{if } k \notin \underline{\sigma}, \underline{\tau} \\ \tau(k) & \text{if } k \in \underline{\tau}. \end{cases}$$

while

$$(\tau \circ \sigma)(k) = \begin{cases} \tau(k) & \text{if } k \notin \underline{\sigma} \\ \sigma(k) & \text{if } k \in \underline{\sigma} \end{cases}$$

$$= \begin{cases} \tau(k) & \text{if } k \in \underline{\tau} \\ k & \text{if } k \notin \underline{\sigma}, \underline{\tau} \\ \sigma(k) & \text{if } k \in \underline{\sigma} \end{cases}$$

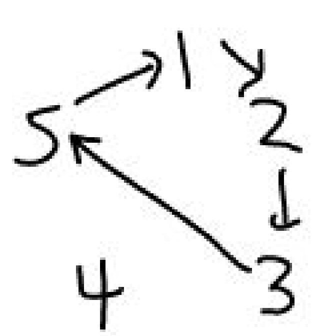
So $\tau \circ \sigma = \sigma \circ \tau$. //

Defn Let σ be a cycle. A <u>cycle notation</u> for σ is the expression

$$(a \;\; \sigma(a) \;\; \sigma^2(a) \;\cdots\; \sigma^{|\sigma|-1}(a))$$

for some $a \in \underline{\sigma}$.

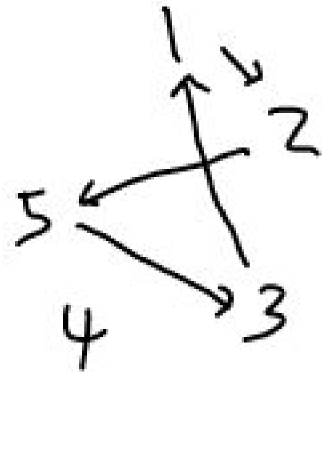
Ex If $\sigma \in S_5$ is pictured as



then the following are all cycle notations for σ:

$$\underset{a=1}{(1\,2\,3\,5)}, \quad \underset{a=2}{(2\,1\,3\,5)}, \quad \underset{a=5}{(5\,1\,2\,3)}, \quad \underset{a=3}{(3\,5\,1\,2)}.$$

Ex If τ is pictured as



, $\underline{\tau} = \underline{\sigma}$, but no cycle notation for τ is a cycle notation for σ.

$$(1\,2\,5\,3), \; (2\,5\,3\,1), \; (5\,3\,1\,2), \; (3\,1\,2\,5)$$

We will write

$$\sigma = (a \ \sigma(a) \cdots \sigma^{|\sigma|-1}(a))$$

for a cycle notation for σ.

Implicitly, we are making identifications between the various cycle notations for σ.

Thm Every element

$$\sigma \in S_n$$

can be written as a product of disjoint cycles, and uniquely, up to reordering.

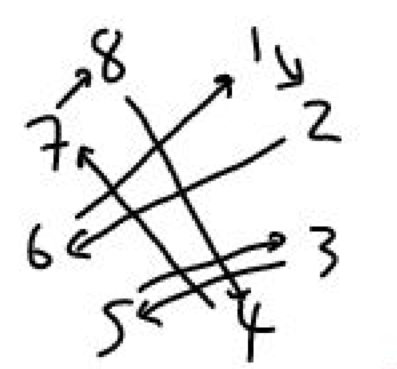
since disjoint cycles commute, the order of composition doesn't matter.

Kind of like prime factorization.

Ex: Let $\sigma \in S_8$ be given by

$$\sigma: \quad \underline{8} \longrightarrow \underline{8}$$
$$1 \longmapsto 2$$
$$2 \longmapsto 6$$
$$3 \longmapsto 5$$
$$4 \longmapsto 7$$
$$5 \longmapsto 3$$
$$6 \longmapsto 1$$
$$7 \longmapsto 8$$
$$8 \longmapsto 4$$

Takes up space



hard to read

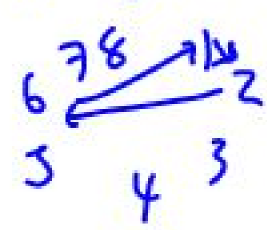
Then $\sigma = (784) \circ (126) \circ (35)$

$= (126)(784)(35)$

$= (126)(35)(784)$

etc.

identifying a cycle in S_8 w/ its cycle notation. So (126) IS the element of S_8 pictured as



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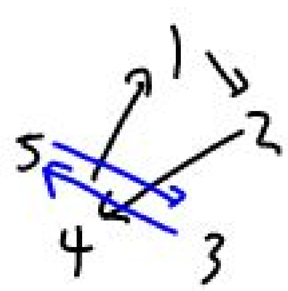
dropping the composition symbol "$\circ$" for brevity.

Defn For $\sigma \in S_n$, a cycle notation for σ is an expression

$$\sigma = \sigma_1 \cdots \sigma_k$$

where each σ_i is a cycle, and each pair σ_i, σ_j is disjoint when $i \neq j$.

Ex If σ is



then the following are cycle notations for σ:

$(124)(35)$	$(35)(124)$
$(124)(53)$	$(53)(124)$
$(412)(35)$	$(35)(412)$
$(412)(53)$	$(53)(412)$
$(241)(53)$	$(53)(241)$
$(241)(35)$	$(35)(241)$.

All these represent the same σ.

Pf. For $\sigma \in S_n$, let

$\{\mathcal{O}_a\}$ be the set of

orbits of σ's action on $\underline{n}$.

$\forall$ $\mathcal{O}_a \in \underline{n}/\langle\sigma\rangle$, choose $a \in \mathcal{O}_a$,

and set

$$\sigma_a := (a \;\; \sigma(a) \; \cdots \; \sigma^{|\mathcal{O}_a|-1}(a))$$

be a cycle. Then by

definition,

$$\sigma = \prod_{\mathcal{O}_a} \sigma_a \, .$$

This is because

$$\prod_{\mathcal{O}_a} \sigma_a \, (k) = \sigma(k)$$

by definition. Note also I've

written $\prod_{\mathcal{O}_a} \sigma_a = \sigma_a \circ \sigma_b \circ \cdots \sigma_z$

w/o specifying an order. This

is because each σ_a, σ_b is

disjoint (by disjointness of orbits)

and hence commute. (Ie, order

doesn't matter.)

As for uniqueness: If someone else

were to write

$$\sigma = \prod \tau_i = \tau_1 \tau_2 \cdots \tau_k$$

for $\{\tau_i\}$ a collection of disjoint

cycles, we note that

$$\{\underline{\tau_i}\} = \underline{n}/\langle\sigma\rangle.$$

$\Rightarrow \forall i$, $\exists!$ σ_a s.t. $\underline{\sigma_a} = \underline{\tau_i}$. Writing $\tau_i = (b_0 \, b_1 \cdots b_{|\tau_i|-1})$

we see $\sigma(b_i) = b_{i+1}$, and we're

done. //

Exer

Let

$$\sigma = (12)(34)$$

$$\tau = (123).$$

Then compute

$$\tau\sigma\tau^{-1}$$

$$\sigma\tau\sigma^{-1}.$$

Pf Note inverse of a cycle is just reading cycle backward, so

$$\tau^{-1} = (321)$$

$$\sigma^{-1} = (21)(43) = \sigma.$$

We see

$$\tau\sigma\tau^{-1} = (123)\circ(12)(34)\circ(321)$$

$$= (14)(23)$$

$$\sigma\tau\sigma^{-1} = (12)(34)\circ(123)\circ(12)(34)$$

$$= (3)(421)$$

$$= (421).$$

Note $(\text{blah})\,\sigma\,(\text{blah})^{-1}$ has same cycle shape as σ. This will help us classify conjugacy classes of S_n.