Wed, Sept 24, 2014

Last time we proved:

Prop'n Let $\phi : G \to G'$ be a surjective group homomorphism. Then $\exists$ an isomorphism

$$G/\ker(\phi) \xrightarrow{\cong} G'.$$

Now we prove

First Isomorphism Theorem

Let $\phi : G \to G'$ be a group homomorphism. Then $\exists$ an isomorphism

$$G/\ker(\phi) \xrightarrow{\cong} \mathrm{im}(\phi).$$

Pf By definition of image, the homomorphism $\phi : G \to G'$ factors as follows:

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, "\underline{\phi}"'] & & G' \\ & \mathrm{im}(\phi) \arrow[ur, "j"'] & \end{tikzcd}.$$

Here, j is the inclusion of $\mathrm{im}(\phi)$ into G'. (It's an injective group homomorphism.) $\underline{\phi}$ is the "same" function as ϕ, but has a different target/codomain. So we see $\phi = j \circ \underline{\phi}$.

By definition of image, Φ is a surjection. Hence the proposition says

$$G/_{\mathrm{Ker}(\Phi)} \cong \mathrm{im}(\phi).$$

But $\mathrm{Ker}(\Phi) = \mathrm{Ker}(\phi)$, since

$$j(1_{\mathrm{im}(\phi)}) = 1_{G'}. \quad //$$

Some leftovers from last class.

We only proved that

$$g\,\mathrm{Ker}(\phi)g^{-1} \subset \mathrm{Ker}\,\phi \quad \forall g$$

when trying to show $\mathrm{Ker}\,\phi$ <u>is</u> normal. But how do we show

$$g\,\mathrm{Ker}(\phi)\,g^{-1} = \mathrm{Ker}\,\phi?$$

Exer Let $H \subset G$ be a subgroup. Show

"$\underbrace{\forall g \in G}$, $gHg^{-1} \subset H$"

very important!

implies

"$\forall g \in G$, $gHg^{-1} = H$."

Pf We need to show that $\forall g \in G$, $H \subset gHg^{-1}$.

So fix $h \in H$.

Let $g' = g^{-1}$.

By hypothesis,

$$g'H(g')^{-1} \subset H, \text{ so}$$

$$g'h(g')^{-1} = h'$$

for some $h' \in H$. Then

$$h = gh'g^{-1}$$

← this shows $h \in gHg^{-1}$.

since

$$\begin{aligned} gh'g^{-1} &= g(g'h(g')^{-1})g^{-1} \\ &= gg^{-1}hgg^{-1} \\ &= h. \end{aligned}$$

//

Cor If H_1, H_2 are normal subgrps of G, so is $H_1 \cap H_2$.

Note that if H_1, H_2 are just subgrps of G, then $H_1 \cap H_2$ is also a subgrp of G.

Pf: $1_G \in H_1$ and H_2 since both are subgrps

Hence $1_G \in H_1 \cap H_2$.

If g and g' are in $H_1 \cap H_2$, then

$gg' \in H_1$ since H_1 is a subgp

$gg' \in H_2$ since H_2 is a subgp

Hence $gg' \in H_1 \cap H_2$.

Likewise for inverses //

Pf Let $h \in H_1 \cap H_2$. Then $\forall g \in G$,

- $ghg^{-1} \in H_1$ since H_1 is normal
- $ghg^{-1} \in H_2$ since H_2 is normal.

Hence $ghg^{-1} \in H_1 \cap H_2$.

$\Rightarrow \forall g \in G$, $gH_1 \cap H_2 g^{-1} \subset H_1 \cap H_2$. //

Now lets say you're given a group G, and some arbitrary collection

$$I$$

of elements in G.

(I isn't a subgroup or anything necessarily; it's just some random list of elements of G.)

Can you find a <u>normal</u> subgroup of G containing I?

Well, G itself is a normal subgroup of G. And it certainly contains I. Can we get something smaller?

Yes.

Consider the intersection

$$\bigcap_{\{H \subset G \text{ s.t. } H \text{ is normal and } H \text{ contains } I\}} H \quad .$$

The set $\{H \subset G \text{ s.t. } \cdots\}$ is NOT empty since G is in it. So we get some normal subgroup of G via this intersection (by corollary). Nice construction.

Defn Let $H \subset G$ be a subgrp. The <u>index</u> of H in G is the number of elements in

possibly infinite!

$$G/H.$$

It is written

$$[G:H].$$

Exer Suppose $K \subset H \subset G$ are subgrps. (H is a subgp of G, K is a subgp of H. Note this also means K is a subgp of G.) Show

$$[G:K] = [G:H][H:K].$$

Pf In the proof of Lagrange's theorem, we saw that

$$G = \coprod_{O_g \in G/H} O_g.$$

ie, that G is a disjoint union of the orbits of H's action on G.

Moreover, $|O_g| = |H| \;\forall g \in G$.

ie, all orbits have the same size.

Hence

$$|G| = n \cdot |H|$$

where n <u>is</u> the number of distinct orbits.

i.e., $n = [G:H]$. Hence

$$[G:K] = |G|/|K| = |G|/|H| \cdot |H|/|K| = [G:H][H:K]. \quad //$$

[just dividing the # $|G|$ by another number, $|K|$!

This one brings many ideas together:

Exer Let

note $\det(A^TA) = \det(I) = 1$; and $\det(A^T) = \det(A)$. Hence $\det(A) = \pm 1$.

$$O_n(\mathbb{R}) := \{ n\times n \text{ real matrices } A \text{ such that } A^TA = I \}$$

and

$$SO_n(\mathbb{R}) := \{ A \in O_n(\mathbb{R}) \text{ s.t. } \det(A) = 1 \}.$$

Compute

$$[O_n(\mathbb{R}) : SO_n(\mathbb{R})].$$

Pf The index is <u>two</u>. Why?

Consider the homomorphism

$$\det : O_n(\mathbb{R}) \longrightarrow \mathbb{R}^\times$$
$$A \longmapsto \det(A).$$

Since the unit of $\mathbb{R}^\times$ is $1 \in \mathbb{R}^\times$, the kernel of det is $SO_n(\mathbb{R})$.

On the other hand,

$$\text{image}(\det) = \{+1, -1\} \subset \mathbb{R}^\times.$$

Since

$$\begin{aligned}
[O_n(\mathbb{R}) : SO_n(\mathbb{R})] &:= \left| O_n(\mathbb{R}) / SO_n(\mathbb{R}) \right| && \text{by defn of index} \\
&= \left| O_n(\mathbb{R}) / \ker(\det) \right| \\
&= \left| \text{image}(\det) \right| && 1^{st} \cong \text{ theorem} \\
&= \left| \{1, -1\} \right| \\
&= 2. \quad //
\end{aligned}$$

Next time: cycle
notation for symmetric
group elements!