Monday, Sept 22, 2014

Let's finish proof.

Thm $H \subset G$ normal.
Then
$$G/H \times G/H \longrightarrow G/H$$
$$(Hg_1, Hg_2) \longmapsto Hg_1g_2$$
is well-defined, and gives G/H a group structure.

Pf

$$H_{g_1'g_2'} = \{h\, g_1'g_2'\}$$
$$= \{h\, h_1g_1\, h_2g_2\}$$
$$= \{h\, h_1g_1h_2g_1^{-1}g_1g_2\}$$
$$= \{h\, h_1h_3\, g_1g_2\}$$
$$\subset H_{g_1g_2}.$$

These are equiv classes, so
$$H_{g_1'g_2'} \subset H_{g_1g_2} \Rightarrow H_{g_1'g_2'} = H_{g_1g_2}.$$

Group?
$$Hg_1(Hg_2Hg_3) = Hg_1\, H_{g_2g_3}$$
$$= H_{g_1(g_2g_3)}$$
$$= H_{(g_1g_2)g_3}$$
$$= H_{g_1g_2}Hg_3$$
$$= (Hg_1Hg_2)Hg_3 \Longrightarrow \text{assoc.}$$

$$Hg_1H_{1_G} = H_{g_1 1_G} = Hg_1,$$
$$H_{1_G}Hg_1 = H_{1_Gg_1} = Hg_1.$$
$\Longrightarrow$ identity H_{1_G}.

You can do inverses.

<u>Prop'n</u> Let $H \subset G$ be normal. The map

$$q : G \longrightarrow G/H$$
$$g \longmapsto Hg$$

(1) is a group homomorphism.

(2) is a surjection

(3) has kernel q.

<u>Pf</u>

(1) $q(g_1 g_2) = Hg_1 g_2$
$= Hg_1 Hg_2$
$= q(g_1) q(g_2)$

(2) $\forall\ Hg \in G/H$,
$Hg = q(g)$.

(3) $q(g) = 1_{G/H} \iff q(g) = H1_G$
$= H$.

But $Hg = H1_G \iff g$ and 1_G are in the same orbit

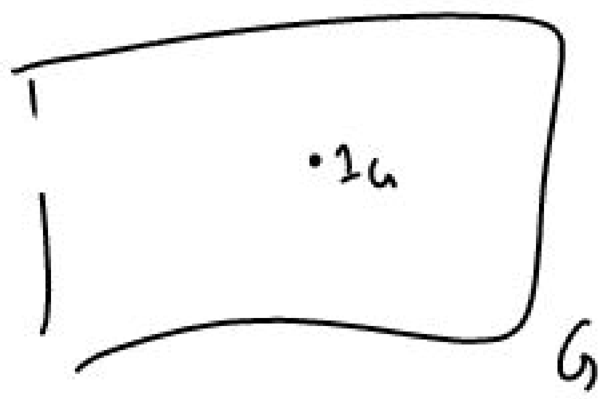
$\iff g = h1_G$ for some $h \in H$

$\iff g \in H$.

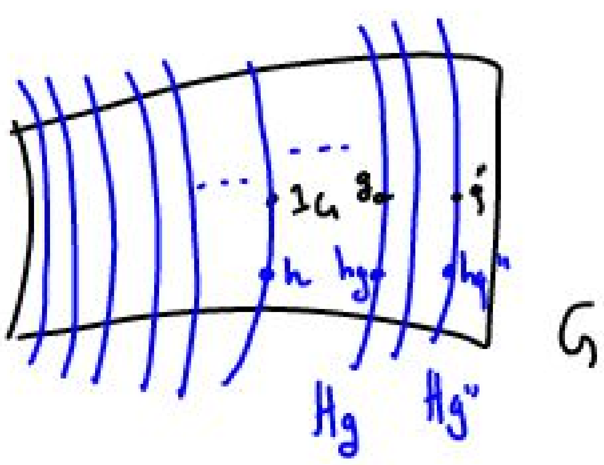
So $q(g) = 1_{G/H} \iff g \in H$. //

What's the picture?

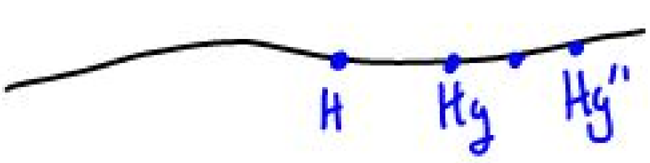
Imagine G as some set



A subgroup $H \subset G$ breaks G up into orbits



And G/H collapses all these orbits:



By the way,

<u>Exer</u>: Let $G \xrightarrow{\phi} G'$

be a group homomorphism.

Then ϕ is injective if and only if

$$\mathrm{Ker}(\phi) = \{1_G\}.$$

<u>Pf</u>:

Injective $\Rightarrow \exists!\ g$ (if any) s.t.

$$\phi(g) = 1_{G'}.$$

Since a gp homom always sends 1_G to $1_{G'}$, $g = 1_G$.

Assume $\mathrm{Ker}(\phi) = \{1_G\}$.

Then $\phi(g_1) = \phi(g_2)$

$$\Rightarrow \phi(g_1)\phi(g_2)^{-1} = 1_{G'}$$

$$\Rightarrow \phi(g_1 g_2^{-1}) = 1_{G'}$$

$$\Rightarrow g_1 g_2^{-1} \in \mathrm{Ker}(\phi)$$

$$\Rightarrow g_1 g_2^{-1} = 1_G$$

$$\Rightarrow g_1 = g_2. \quad //$$

So given any normal H,
the quotient homomorphism q
exhibits H as the kernel of
<u>some</u> grp homomorphism.

Is every kernel of a grp
homomorphism normal?

<u>Prop.</u> Let $\phi: G \longrightarrow G'$
be a grp homomorphism.
Then
$$\mathrm{Ker}(\phi)$$
is normal.

<u>Pf</u> NTS: $\forall \quad h \in \mathrm{Ker}(\phi)$,
$\forall \quad g \in G$,
$$ghg^{-1} \in \mathrm{Ker}(\phi).$$

Well,
$$\begin{aligned}
\phi(ghg^{-1}) &= \phi(g)\,\phi(h)\,\phi(g^{-1}) \\
&= \phi(g)\, 1_{G'}\, \phi(g^{-1}) \\
&= \phi(g)\phi(g^{-1}) \\
&= \phi(gg^{-1}) \\
&= \phi(1_G) \\
&= 1_{G'}.
\end{aligned}$$

So $ghg^{-1} \in \mathrm{Ker}(\phi)$. //

Cor Let $\phi: G \to G'$ be any group homomorphism. Then $\exists$ group isomorphism

$$G/_{\ker(\phi)} \cong \text{image}(\phi).$$

The First Isomorphism Theorem

Pf image$(\phi) \subset G'$ is a subgroup, and by definition, $\phi: G \longrightarrow G'$ factors as

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, "f"'] & & G' \\ & \text{image}(\phi) \arrow[ur, "\text{inclusion}"'] & \end{tikzcd}$$

Also by definition, f is a surjection with the same kernel as ϕ. //

Pf of Prop'n

Given a surjective group homomorphism

$$\phi: G \longrightarrow G',$$

let $H = \ker(\phi)$.

Note that if

$$g_2 \in Hg_1, \quad \phi(g_2) = g_1$$

Because:

$$\phi(g_2) = \phi(hg_1) \quad \text{for some } h \in H$$

$$= \phi(h)\phi(g_1)$$

$$= 1_{G'}\phi(g_1)$$

$$= \phi(g_1).$$

So we have a well-defined map

$$\psi : G/H \longrightarrow G'$$
$$Hg \longmapsto \phi(g).$$

(We showed if $Hg_1 = Hg_2$, then $\phi(g_1) = \phi(g_2)$.)

This is a homomorphism, since

$$\begin{aligned}
\psi(Hg_1 Hg_2) &= \psi(Hg_1 g_2) && \cdot \text{ in } G/H \\
&= \phi(g_1 g_2) && \text{Defn of } \psi \\
&= \phi(g_1)\phi(g_2) && \phi \text{ is a homom} \\
&= \psi(Hg_1)\psi(Hg_2) && \text{Defn of } \psi.
\end{aligned}$$

It is an injection since

$$\begin{aligned}
\psi(Hg_1) = 1_{G'} &\iff \psi(g_1) = 1_{G'} \\
&\iff g_1 \in H \\
&\iff Hg_1 = H1_G, \text{ the unit of } G/H.
\end{aligned}$$

It is a surjection since ϕ is — $\forall\ g' \in G', \exists$ some $g \in G$ st $\phi(g) = g'$ so

$$\psi(Hg) = g'! \quad //$$

Defn Given

$H \subset G$ a subgrp,

the *index* of H in G

is the number

$$|G|/|H| = \#\text{ of cosets } Hg$$

$$= \#\text{ of orbits } \mathcal{O}_g$$

and is written

$$[G:H].$$

Exer Let $O_n(\mathbb{R})$ be

the grp of $n \times n$ matrices A s.t.

$$A^T A = I.$$

Let $SO_n(\mathbb{R}) \subset O_n(\mathbb{R})$ be those

A for which $\det(A) = 1$. Show

$$[O_n(\mathbb{R}) : SO_n(\mathbb{R})] = 2.$$

ie. show $SO_n(\mathbb{R})$ is an index 2 subgrp

of $O_n(\mathbb{R})$.