$\boxed{\text{Wed, Sept 17 2014}}$

Let's finish:

Thm Every $w \in \text{Word}(S)$ has a unique reduction

Pf Induction on length, ℓ.

$\ell = 0$ obvious

$\ell = 1$ obvious

Assume that $\forall$ w' with length $\leq \ell - 1$, the set

$$\{\text{reductions of } w'\}$$

has only one element. We must prove this is true $\forall$ words w of length ℓ.

- If w is already reduced, then no other word can be obtained from w by cancelling. Hence

  $\{\text{reductions of } w\}$ has only one element — w itself.

- Otherwise, $\exists$ an appearance of

$$aa^{-1} \quad \text{or} \quad a^{-1}a$$

somewhere in w.

Let's fix a single such appearance,

$$w = \cdots \quad \underline{aa^{-1}} \cdots$$

which we've underlined.

$\cdots \underline{aa^{-1}} \cdots \underline{b^{-1}b} \cdots$

① $\cdots \cancel{aa^{-1}} \cdots b^{-1}b \cdots$ same result! $\cdots aa^{-1} \cdots \cancel{bb^{-1}} \cdots$ ①

② $\cdots \cancel{aa^{-1}} \cdots \cancel{bb^{-1}} \cdots$ ②

since

This is an equality,

since if $\underline{aa^{-1}}$ is cancelled in step N, you'd get same reduction by first having cancelled $\underline{aa^{-1}}$, then performing steps 1 through $N-1$.

Let's consider:

$$\left\{\begin{array}{l}\text{reductions of } w \\ \text{obtained by cancelling} \\ \underline{aa^{-1}} \text{ at the first step}\end{array}\right\} \overset{①}{=} \left\{\begin{array}{l}\text{reduction of } w \\ \text{obtained by cancelling} \\ \underline{aa^{-1}} \text{ at some step}\end{array}\right\}$$

② this is either an equality, or the set is empty.

$$\{\text{reductions of } w \text{ obtained by } \underline{\text{never}} \text{ cancelling } \underline{aa^{-1}} \text{ itself}\}.$$

and every reduction of w is in $\underline{\text{one}}$ of these sets!

since NOT cancelling $\underline{aa^{-1}}$ means you had to cancel

$\underline{a}\underline{a}^{-1}$ or $\underline{aa^{-1}}a$ like this

like this at some stage.

But $\cdots \cancel{aa}a^{-1} \cdots = \cdots a\cancel{aa^{-1}} \cdots$ AND $\cdots aa^{-1}a \cdots = \cdots \cancel{aa^{-1}}a \cdots$ Then

ex: $w = \underline{aa^{-1}}a$

$a\cancel{a^{-1}a} \rightsquigarrow a$

never cancels $\underline{aa^{-1}}$.

① and ② tell us that every reduction of w can be obtained by first cancelling $\underline{aa^{-1}}$. But

$$w' = \cdots \cancel{\underline{aa^{-1}}} \cdots$$

is a word of length $< l$!

Moreover, any reduction of w obtained by <u>first</u> cancelling $\underline{aa^{-1}}$ is a reduction of w'.

Hence

$$\left\{ \begin{array}{c} \text{reductions of } w \text{ obtained} \\ \text{by } \underline{\text{first}} \text{ cancelling} \\ \underline{aa^{-1}} \end{array} \right\}$$

$$\|$$

$$\{\text{reductions of } w'\}$$

$$\|$$

a set w/ one element. //

Ex If $S=\emptyset$,

Word(S)

is a set with one element — the empty word, i.e., the word of length zero.

Ex If $S=\emptyset$,

$$\text{Free}(S) = \left\{ \begin{matrix} \text{reduced words} \\ \text{in } \underline{S} = \emptyset \end{matrix} \right\}$$

= the set containing the empty word.

So Free(S) is a group with one element when $S=\emptyset$!

Ex Via your homework, for all functions

$$\underset{\text{a set}}{S} \longrightarrow \underset{\text{a group}}{G}$$

$\exists$ a group homomorphism

$$F(S) \longrightarrow G.$$

When $S=\emptyset$, $\exists!$ function

$$\emptyset \longrightarrow G.$$

What group homomorphism is

$$F(\emptyset) \longrightarrow G?$$

It sends the empty word to 1_G.

Quotient groups

Let $H \subset G$ be a subgroup
When can the orbit set

$$G/H$$

be given a group structure?

Some special terminology:

Defn Let $H \subset G$ be a subgroup.
Let $g \in G$. We define

$$Hg := \{ y = hg \mid h \in H \}$$
$$\subset G,$$

and call Hg a
right coset of H.

Rmk Hg is the orbit of g
with respect to the action
of H on G. So

$$Hg = \mathcal{O}_g.$$

So the question: is there
a natural group operation of
the set of cosets of H?

A candidate:

$$G/H \times G/H \longrightarrow G/H$$
$$(Hg_1, Hg_2) \mapsto Hg_1g_2$$

Is this map well-defined?

No, in general.

But it is well-defined in special cases:

Defn A subgroup $H \subset G$ is called normal if $\forall\, g \in G$,

$$\{ghg^{-1} \mid h \in H\} = H.$$

The lefthand side is also written

$$gHg^{-1}.$$

ie, H is normal iff

$$gHg^{-1} = H.$$

We'll next prove

<u>**Thm**</u> If $H \subset G$ is normal, the operation

$$G/H \times G/H \to G/H$$
$$(Hg_1, Hg_2) \mapsto Hg_1g_2$$

is well-defined, and makes G/H into a group.

Let's see some examples.

<u>**Ex**</u> $H = \{id_G\} \subset G$.

H is normal, since $\forall\, g \in G$,

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$
$$= \{g\, id_G\, g^{-1}\}$$
$$= \{id_G\}$$
$$= H.$$

But G/H isn't a special new group, b/c $\exists$ an isomorphism

$$G/H \longrightarrow G\ .$$
$$Hg \longmapsto g$$

Next time:

$H = n\mathbb{Z}$,

$G/H = ?$