Mon, Sept 15, 2014

Last week: Lagrange's thm ← about finite groups
+
Free groups. ← never finite.

I want to simplify both their presentations.

Defn Let X be a set.
An equivalence relation on X is a subset
$$R \subset X \times X$$
satisfying:
(i) $(x,x) \in R \quad \forall x \in X$
(ii) $(x,y) \in R \Rightarrow (y,x) \in R$
(iii) $(x,y) \in R$ and $(y,z) \in R$
$\Longrightarrow (x,z) \in R.$

We will say $x \sim y$ if $(x,y) \in R$.

Exer Let G act on a set X.
Say
$$x \sim y \iff y = gx \text{ for some } g \in G.$$
This is an equivalence relation:
(i) $x = 1_G x$, so $x \sim x$.
(ii) $x \sim y \Rightarrow y = gx$ for some g $\Rightarrow x = g^{-1}y \Rightarrow y \sim x$.
(iii) $y \sim z \Rightarrow z = g'y \Rightarrow z = g'gy \Rightarrow z \sim x$. //

Then $\forall x$, $\mathcal{O}_x$ is just the equivalence class of x. (i.e., $\mathcal{O}_x = \{y \mid y \sim x\}$.)

And X/G is the set of equivalence classes.

Here's another example.

Let S be a set, and set

$$S' = \{x^{-1}\}_{x \in S},$$

$$\underline{S} = S \cup S'.$$

Later we'll prove:

__Thm__ Every $w \in \mathrm{Word}(\underline{S})$ has a unique reduction.

Assuming this

__Exer__ Say $w_1 \sim w_2$ if w_1 and w_2 have the same reduction. Show this is an equivalence relation.

__Rmk__ So if $w_1 \rightsquigarrow w_2$ via cancellation, then $w_1 \sim w_2$. (Not nec. conversely.)

Pf (of Ex$\circ$).

(i) $w \sim w$ obviously: reduction(w) = reduction(w).

(ii) $w_1 \sim w_2 \Rightarrow w_2 \sim w_1$

obviously, since reduction(w_1) = reduction(w_2)

$\Rightarrow$ reduction(w_2) = reduction(w_1).

(iii) $w_1 \sim w_2$, $w_2 \sim w_3 \Rightarrow w_1 \sim w_3$

since

reduction(w_1) = reduction(w_2), reduction(w_2) = reduction(w_3) $\Rightarrow$ reduction(w_1) = reduction(w_3).

(All follows from uniqueness of reduction, and the fact that equality is an equivalence relation)

Prop. $\exists$ bijection

$$F(S) \longrightarrow \{\text{equivalence classes of words in Word}(S)\}.$$

reduced words in S.

Pf.

Send $w \longmapsto [w]$. Ie, send w to its equivalence class.

Any equivalence class has a unique element of shortest length — the (common) reduction of any $w \in [w]$. This defines inverse map. //

Prop'n The operation

$$\begin{Bmatrix}\text{equiv}\\ \text{classes}\\ \text{of}\\ \text{words}\end{Bmatrix} \times \begin{Bmatrix}\text{equiv}\\ \text{classes}\\ \text{of}\\ \text{words}\end{Bmatrix} \xrightarrow{\text{Concatenate}} \begin{Bmatrix}\text{equiv}\\ \text{classes}\\ \text{of}\\ \text{words}\end{Bmatrix}$$

$$([w_1], [w_2]) \longmapsto [w_1 w_2]$$

is well-defined.

Pf Let r_1 and r_2 be reductions of w_1', w_2', respectively. Then note

$r_1 r_2$ can be obtained from $w_1' w_2'$ via cancellations.

(Just apply cancellations to the w_1' part of the word, then to the w_2' part of the word.) Hence for any

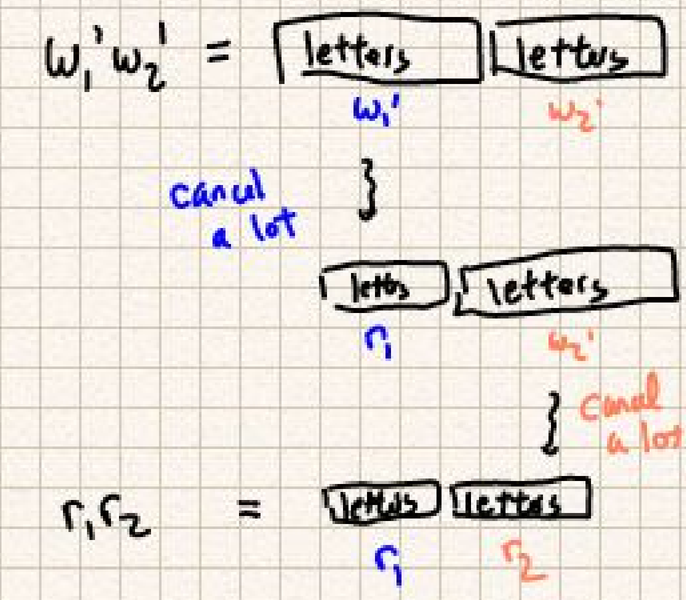
$$w_1' \in [w_1], \quad w_2' \in [w_2],$$

we have

$$w_1' w_2' \sim r_1 r_2.$$

I.e.,

$$[w_1' w_2'] = [r_1 r_2] \qquad //$$



So regardless of which representatives $w_1' \in [w_1]$, $w_2' \in [w_2]$ we choose, the equiv class of $w_1' w_2'$ is unchanged.

Cor The free group operation from last time

$$\begin{aligned} F(S)\times F(S) &\longrightarrow F(S) \\ (r_1, r_2) &\longmapsto \text{reduction}(r_1 r_2) \end{aligned}$$

is associative.

Pf

$$\begin{tikzcd}
(r_1,r_2) & & \text{reduction}(r_1 r_2) \\
F(S)\times F(S) \arrow[rr] \arrow[d, "\text{biject.}"'] & & F(S) \\
\begin{Bmatrix}\text{equiv}\\ \text{classes}\end{Bmatrix} \times \begin{Bmatrix}\text{equiv}\\ \text{classes}\end{Bmatrix} \arrow[rr, "{[\text{concatenate}]}"] & & \begin{Bmatrix}\text{equiv}\\ \text{classes}\end{Bmatrix} \arrow[u] \\
([r_1],[r_2]) \arrow[rr, mapsto] & & [r_1 r_2]
\end{tikzcd}$$

so we just need to show the operation $([r_1],[r_2]) \mapsto [r_1 r_2]$ is associative. Well,

$$([r_1][r_2])[r_3] = [r_1 r_2][r_3] = [(r_1 r_2) r_3] = [r_1 (r_2 r_3)] = [r_1][r_2 r_3] = [r_1]([r_2][r_3]).$$

by associativity of concatenating ordinary words.

<u>Prop'n</u> If w' and w'' are reductions of w, then
$$w' = w''!$$

<u>Pf</u> Induction on length w of a word. (Note $w \rightsquigarrow u \Rightarrow \text{length}(u) < \text{length}(w)$).

$l=0$: empty word is reduced

$l=1$: No a, a^{-1} can occur adjacently since there's only one element in the word. So $l=1 \Rightarrow$ word is reduced.

Suppose we've shown every word of length $l-1$ has unique reduction. Prove the same for l.

If w has length l and is reduced, done.

If not, $\exists$ aa^{-1} or $a^{-1}a$ somewhere. Potentially many of them!

<u>Ex</u> $a^{-1}a\,a^{-1}a\,a^{-1}a = w$, length 6.

Pick one of them.

$\cdots \ \underline{a^{-1}a} \cdots$.

A reduction of ω can be achieved by:

(i) cancelling $\underline{a^{-1}a}$ at some stage.

(ii) never cancelling $\underline{a^{-1}a}$.

(ii) only happens if

(*) $\cdots \underline{a^{-1}a}\ a^{-1} \cdots$ appears at some stage, and we take

$$\cdots \underline{a^{-1}}\underbrace{a\, a^{-1}}_{\text{cancel}} \cdots \rightsquigarrow \cdots \underline{a^{-1}} \cdots$$

or

(**) $\cdots a\, \underline{a^{-1}a} \cdots$ appears and we take

$$\cdots \underbrace{a a^{-1}}_{\text{cancel}}\underline{a} \cdots \rightsquigarrow \cdots \underline{a} \cdots$$

In (*), cancelling $\underline{a^{-1}a}\ a^{-1}$

by $\underbrace{\underline{a^{-1}a}}\,a^{-1}$ or $\underline{a^{-1}}\underbrace{\underline{a}\,a^{-1}}$ produces the same word. Likewise for (**).

So we can assume $\underline{a^{-1}a}$ is reduced at some point.

(Any reduction achieved via (ii) can be achieved via (i).)

So we have a reduction

$$w = \cdots \quad \underline{aa^{-1}} \cdots \underline{b^{-1}b} \cdots c^{-1}c$$

$\wr$ Step one

$$w_1 = \cdots \quad \underline{aa^{-1}} \cdots \cdots \cdots \quad \underline{c^{-1}c}$$

$\wr$

$\vdots$

$\wr$ Step n

w_n reduced.

Well, we get the same reduction if we cancel $\underline{aa^{-1}}$ in step 1, or in some other step. //

<u>Defn</u> Let S be a set. Then the <u>free group</u> $F(S)$ on S is

$$(F(S), m)$$

where

- $F(S)$ is the set of reduced words in $\underline{S} = S \cup S'$.
- $m : F(S) \times F(S) \to F(S)$ sends (w_1, w_2) to the reduction of $w_1 w_2$.