Wed, Sept 10, 2014

Last time:

Defn A group action of G on a set X is a function ϕ which sends each $g \in G$ to a map $\phi_g : X \to X$. This map must be a bijection, and we require

dependence on g.

$gh \in G$, so what's the map associated to it?

$$\phi_{gh} = \phi_g \circ \phi_h.$$

Composite of fxn associated to g, and to h

Exer A group action determines a map of sets

$$G \times X \to X$$

where we will write the value of (g, x) as gx.

The map satisfies

(a) $1_G x = x$

(b) $(gh)x = g(hx)$.

Conversely, any map $G \times X \to X$ satisfying (a), (b) determines a group action.

Defn Let G be a group. We let

$$|G| \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$$

be the number of elements in G. We call $|G|$ the <u>order</u> of G.

Defn Fix $g \in G$.

Consider the set

$$\{\dots, \underset{\underset{g^{-2}}{\shortparallel}}{g^{-1}g^{-1}}, g^{-1}, id_G, g, \underset{\underset{g^2}{\shortparallel}}{g \cdot g}, \underset{\underset{g^3}{\shortparallel}}{g \cdot g \cdot g}, \dots\}$$

it's a subgroup of G, since

$$g^a \cdot g^b = g^{a+b}.$$

We denote this subgroup by $\langle g \rangle$, and we define the <u>order of g</u> to be

$$|\langle g \rangle|.$$

Ex
- $1_G \in G$ has order $\underline{1}$.
- $n \in \mathbb{Z}$, $n \neq 0$ has infinite order.
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{R})$ has order 2.

$$g^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{GL_2(\mathbb{R})}$$

So $\{\dots, g^{-1}, 1, g, \dots\}$

$= \{1, g\}$.

Pf Given
$$\phi : G \to \mathrm{Aut}_{Set}(X)$$
let $\phi(g) =: \phi_g$.

Then let
$$G \times X \longrightarrow X$$
be
$$(g, x) \longmapsto \phi_g(x).$$

(a) Then $\phi_{1_G} = id_X$ (since ϕ is a homom)

so $(1, x) \longmapsto \phi_1(x) = id_X(x)$
$= x$.

(b) $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$ since ϕ is a grp homom.

Hence
$$\phi_{g_1 g_2} = \phi_{g_1} \circ \phi_{g_2} \quad \text{as fxns } X \to X$$

meaning
$$\phi_{g_1 g_2}(x) = \phi_{g_1} \circ \phi_{g_2}(x) \quad \forall x.$$

By notation,
$$(g_1 g_2)(x) = \phi_{g_1}(g_2 x)$$
$$= g_1(g_2 x).$$

Conversely. If we're given a map $G \times X \to X$ satisfying (a), (b), restrict the map to the set $\{g\} \times X \subset G \times X$.

The map $\{g\} \times X \to X$ can be identified with a map
$$\psi_g : X \cong \{g\} \times X \longrightarrow X$$
$$x \longmapsto (g, x) \longmapsto x.$$

ψ_g is a bijection because of (a) and (b). First, look at

$$\psi_{1_G} : X \cong \{1_G\} \times X \to X$$
$$x \mapsto (1_G, x) \mapsto 1_G \cdot x$$

By (a), $1_G \cdot x = x$, so

$$\psi_{1_G}(x) = x.$$

This means ψ_{1_G} is the identity bijection, $\psi_{1_G} = \mathrm{id}_X$. ← equal as elements in the set of maps from X to X.

Next, note ψ_g is a bijection $\forall g$. This is because

injection: $\psi_g(x) = \psi_g(y)$

$$\Rightarrow \quad gx = gy \qquad \text{Notation}$$

$$\Rightarrow \quad g^{-1}(gx) = g^{-1}(gy) \qquad \text{Since } G \times X \to X \text{ is given.}$$

$$\Rightarrow \quad (g^{-1}g)x = (g^{-1}g)y \quad (b) \qquad (x' = x \Rightarrow hx' = hx \ \forall h \in G)$$

$$\Rightarrow \quad 1_G x = 1_G y$$

$$\Rightarrow \quad x = y \quad (a).$$

surjection:
If $x \in X$, let $y = g^{-1}x$.
Then
$$\psi_g(y) = g(g^{-1}x) = (gg^{-1})x = x.$$

So $\psi_g \in \mathrm{Aut}_{Set}(X)$

$\forall g \in G$.

Finally, $g \mapsto \psi_g$ is a homomorphism since

$$g_1 g_2 \mapsto \psi_{g_1 \cdot g_2},$$

and
$$\begin{aligned}
\psi_{g_1 g_2}(x) &= (g_1 g_2)x \\
&= g_1(g_2 x) \quad (b) \\
&= \psi_{g_1}(g_2 x) \\
&= \psi_{g_1}(\psi_{g_2}(x)) \\
&= \psi_{g_1} \circ \psi_{g_2}(x) \ \forall x.
\end{aligned}$$

$\rightarrow \psi_{g_1 g_2} = \psi_{g_1} \circ \psi_{g_2}$. //

(How do you show that two functions $f, g : A \to B$ are the same? Show $f(a) = g(a)$ for all $a \in A$. That's the definition of $f = g$.)

So you may find it healthier to think of a group action as a map
$$G \times X \to X$$
satisfying (a), (b) rather than as a homomorphism
$$G \to \mathrm{Aut}_{Set}(X).$$

Def. Let G act on a set X. Then $\forall\, x \in X$, the orbit of x is the set

$$\mathcal{O}_x = \{ y \in X \mid y = gx \text{ for some } g \}$$

Ex · $G = S^1$, $X = \mathbb{C}$,

$$G \times X \longrightarrow X$$
$$(e^{i\theta}, z) \longmapsto z \times e^{i\theta}$$

Then $\mathcal{O}_z = \{ w \mid w = z \times e^{i\theta} \text{ for some } \theta \}$

= circle of radius $|z|$.

· Let $G \longrightarrow \mathrm{Aut}_{Set}(X)$ be $g \longmapsto id_X$.

(The trivial action.) Then

$$\mathcal{O}_x = \{ y \mid y = id_X(x) \}$$
$$= \{x\}.$$

Rmk Let $P(X)$ be the power set of X. It's the set of all subsets of X. Then a group action determines a map

$$X \longrightarrow P(X)$$
$$x \longmapsto \mathcal{O}_x.$$

certainly won't hit every element of $P(X)$ — for instance, empty subset. But we'll hit some of them.

Defn The orbit set, or orbit space, of a group action is the image of $X \to P(X)$.
We denote it

$$X/G.$$

$\leftarrow$ Like dividing out X by G. If $y = gx$, then $\mathcal{O}_x = \mathcal{O}_y$, so y and x have the same image in $P(x)$ — ie, are sent to the same element in X/G.

Exer

(i) $\forall x \in X$, $x \in \mathcal{O}_x$

(ii) $\mathcal{O}_x = \mathcal{O}_y \iff y = gx$ for some $g \in G$.

Pf (i) $x = 1_G x$, so $x \in \mathcal{O}_x$.

(ii) $\mathcal{O}_x = \mathcal{O}_y \iff y \in \mathcal{O}_x$ by (i)

$\iff y = gx$ for some $g \in G$ (Defn of $\mathcal{O}_x$). //

Prop Let H be a subgroup of G. Then we saw last time H acts on G.
Moreover, $\forall x, y \in G$, $|\mathcal{O}_x| = |\mathcal{O}_y|$.

Pf Let $h = x^{-1}y \in G$. Then we have maps

$$\begin{aligned} \mathcal{O}_x &\to \mathcal{O}_y \\ gx &\mapsto gxh \end{aligned}$$

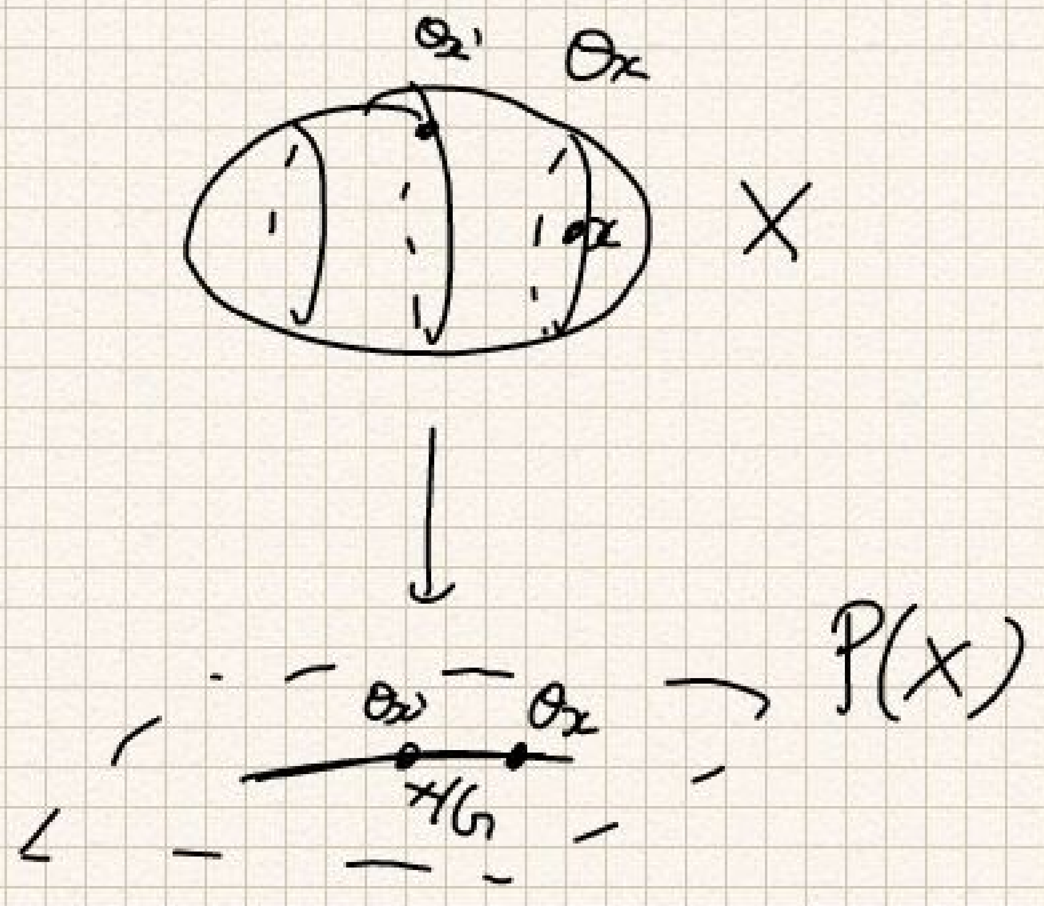
in $\mathcal{O}_y$ since $gxh = gx(x^{-1}y) = gy \in \mathcal{O}_y$.

$$\begin{aligned} \mathcal{O}_x &\leftarrow \mathcal{O}_y \\ gyh^{-1} &\leftarrow gy. \end{aligned}$$

These are inverse to each other, since
$gx \mapsto gxh \mapsto gxhh^{-1} = gx$
$gy \mapsto gyh^{-1} \mapsto gyh^{-1}h = gy$.

Prop.

$$X = \bigcup_{\mathcal{O} \in X/G} \mathcal{O}.$$



Moreover, $\mathcal{O} \neq \mathcal{O}'$

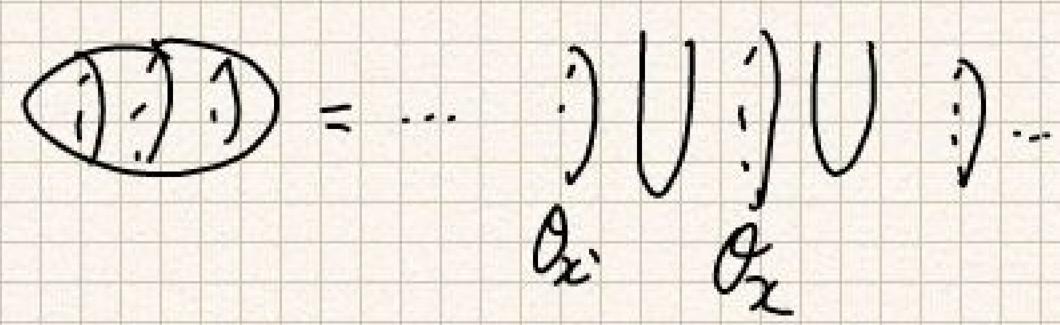
$\Rightarrow \mathcal{O} \cap \mathcal{O}' = \phi.$

Cor/Defn

$$X = \coprod_{\mathcal{O} \in X/G} \mathcal{O}$$

Cor

$$|X| = \sum_{\mathcal{O} \in X/G} |\mathcal{O}|$$

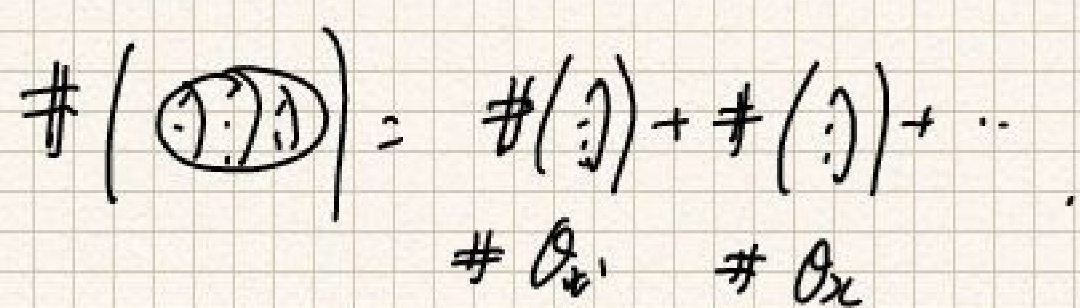


Cor

$$|X| = |X/G| \cdot |\mathcal{O}_x|$$

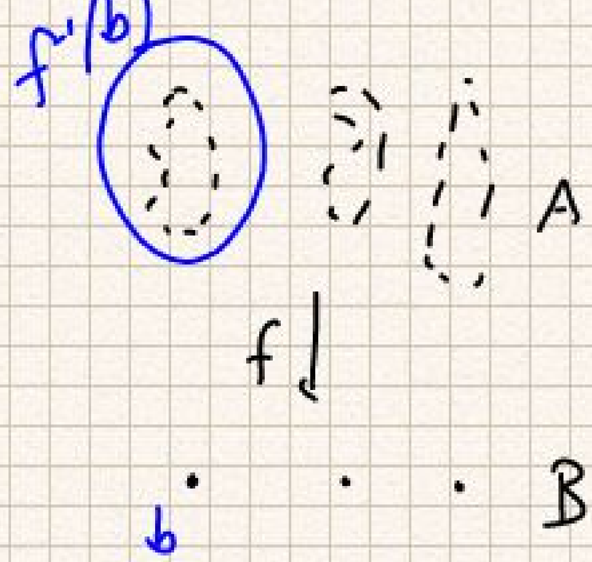
for any $x \in X$.

So



Pf (of Prop). For any map of sets,

$$f : A \longrightarrow B,$$



we know

$$A = \bigcup_{b \in B} f^{-1}(b).$$

And $f^{-1}(b) \cap f^{-1}(b') = \phi$.

Here, $\mathcal{O} = f^{-1}(\mathcal{O})$, so $\mathcal{O} \neq \mathcal{O}' \Rightarrow \mathcal{O} \cap \mathcal{O}' = \phi$. //

Prop'n

$$|\mathcal{O}_{id_G}| = |H|.$$

Pf

$$\mathcal{O}_{id_G} = \{y \mid y = h \cdot id_G \text{ for some } H \ni h\}$$

$$= \{y \mid y = h \text{ for some } h \in H\}$$

$$= H. \quad //$$

We've proven

Thm (Lagrange's theorem)

Let G be finite. Then

$|H|$ divides $|G|$.