Homework Seven

1. Rotational symmetries of the cube

(a) Using the orbit-stabilizer theorem, compute the number of elements in the group of rotations of $\mathbb{R}^3$ that send a perfect cube (centered at the origin) to itself. You might consider looking at faces, and not vertices.
(b) What if, instead of the cube, you consider a regular octahedron (also centered at the origin)? You should note that the regular octahedron can be drawn inside a cube, with each vertex of the octahedron at the center of a face of the cube.

2. Inner product on $\mathbb{R}^n$

Recall that the *dot product* sends a pair $\vec{x}, \vec{y} \in \mathbb{R}^n$ to the real number

$$\vec{x} \cdot \vec{y} := x_1 y_1 + \ldots x_n y_n.$$

Equivalently, if one thinks of $\vec{x}$ and $\vec{y}$ as column vectors—i.e., as $n \times 1$ matrices—we have

$$\vec{x} \cdot \vec{y} = x^T y.$$

We say $\vec{x}$ and $\vec{y}$ are *orthogonal* if $\vec{x} \cdot \vec{y} = 0$. We also note that

$$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}, \qquad \text{and} \qquad (t\vec{x} + \vec{x}') \cdot \vec{y} = t\vec{x} \cdot \vec{y} + \vec{x}' \cdot \vec{y}.$$

Show that the following are equivalent for an $n \times n$ matrix A:

(a) $A^T A = I$. (I.e., $A \in O_n(\mathbb{R})$.)
(b) A preserves the dot product. That is, $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ for every $\vec{x}, \vec{y} \in \mathbb{R}^n$. (Hint: Use that the inner product is a multiplication of a column vector and a row vector.)
(c) The columns of A are mutually orthogonal vectors of unit norm. (Hint: Every entry resulting from a matrix multiplication is a dot product of a row with a column.)

3. Rotations

(a) Let n be odd. Prove that any matrix $A \in SO_n(\mathbb{R})$ has at least one eigenvector with eigenvalue 1. (Hint: Show that $\det(A - I) = \det(I - A)$ by using the fact that $A^T(A - I) = (I - A)^T$.)

(b) Show that any $A \in SO_3(\mathbb{R})$ fixes a non-zero vector v, and A is rotation about this vector. (Hint: A is orthogonal, so it preserves dot products. What can you say about A's effect on the plane orthogonal to v?)

(c) By a *rotation* in $\mathbb{R}^3$, we mean the linear map which rotates $\mathbb{R}^3$ about some line through the origin. Show that the composition of two rotations is again a rotation (even if their axes of rotation do not agree!). Don't try to do this by computational brute force.

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So $SO_3(\mathbb{R})$ is the group of rotations in $\mathbb{R}^3$. (Likewise, you saw last week that $SO_2(\mathbb{R})$ is the group of rotations in $\mathbb{R}^2$, by seeing that $SO_2(\mathbb{R})$ is isomorphic to the circle.) This is a very special situation; in no other dimension does it hold that an element of $SO_n(\mathbb{R})$ is automatically a rotation about some axis.

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(d) Show by example that $SO_4(\mathbb{R})$ has an element which does not fix any vector.

4. Automorphisms of a cyclic group

Let C_n be a finite cyclic group of order n. (So, for instance, it is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.) Let $\phi(n)$ be the number of $1 \leq k \leq n$ for which $gcd(k, n) = 1$. ϕ is called *Euler's totient function*.

(a) Show that $|\operatorname{Aut}(C_n)| = \phi(n)$. (Hint: Show that an automorphism must send a generator to a generator. Then what?)

(b) Show that there are only two isomorphisms types of groups that can be obtained as a semidirect product $\mathbb{Z}/6\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. What are they? (Hint: What are the possible maps from $\mathbb{Z}/2\mathbb{Z}$ to $\operatorname{Aut}(\mathbb{Z}/6\mathbb{Z})$?)