Homework 6

1. Another split short exact sequence

Let $\{\pm 1\} \subset \mathbb{R}^{\times}$ be the subgroup consisting of 1 and -1.

(a) Prove that

$$1 \to SO_n(\mathbb{R}) \to O_n(\mathbb{R}) \to \{\pm 1\} \to 1$$

is a short exact sequence. Here, $SO_n(\mathbb{R}) \to O_n(\mathbb{R})$ is the inclusion.

(b) Exhibit a splitting of the above short exact sequence.

2. $SO_2(\mathbb{R})$ is the circle.

Recall (or convince yourself) that $SO_2(\mathbb{R})$ consists of matrices

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where $a^2 + b^2 = 1$.

(a) Show that $SO_2(\mathbb{R})$ is isomorphic to the group S^1. Here, $S^1 \subset \mathbb{C}^{\times}$ is the subgroup of all complex numbers z such that $|z^2| = 1$.

(b) Prove that $SO_2(\mathbb{R})$ is abelian.

3. The dihedral groups

Recall from class that for any abelian group L, the inversion $\sigma : l \mapsto l^{-1}$ defines a homomorphism

$$\phi : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(L), \qquad [0] \mapsto id_L, \qquad [1] \mapsto \sigma.$$

In particular, for $L = \mathbb{Z}/n\mathbb{Z}$ with $n \geq 2$, this defines a group

$$D_{2n} := \mathbb{Z}/n\mathbb{Z} \rtimes_{\phi} \mathbb{Z}/2\mathbb{Z}.$$

Now let $\langle x, y \rangle \subset O_2(\mathbb{R})$ be the subgroup generated by the matrix x representing rotation by $2\pi/n$ radians, and the matrix y representing reflection about the x-axis.[3]

Prove that $\langle x, y \rangle$ is isomorphic to D_{2n}.

[3] The *subgroup generated by* means the subgroup obtained by taking all elements that are finite products of x, x^{-1}, y, y^{-1}, in any order.