Homework 5

You do not need to turn in the problems marked with an asterisk (*). However, if you do turn it in, we will grade it and you may receive extra credit.

1. Orders revisited

Recall that you proved any subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$.

(a) Let $g \in G$ be an element of finite order n. Show that $g^n = 1_G$. (Hint: any element of G defines a group homomorphism from $\mathbb{Z}$.)
(b) If g is of finite order, show that the order of g is also the smallest number k for which $g^k = 1_G$. (You can use the same trick as above.)
(c) Let G be a finite group. Show that for any $g \in G$, $g^{|G|} = 1_G$.

2. The opposite group

(a) Given a group $G = (G, m)$, define the *opposite group* $G^{\mathrm{op}} = (G, w)$ by the operation

$$w(g, h) := m(h, g).$$

That is, G^{op} as a set is the same set as G, but its multiplication happens in the opposite order. Show that G^{op} is a group.
(b) Show that the map $G \to G^{\mathrm{op}}$ given by $g \mapsto g^{-1}$ is a group isomorphism.

3. Conjugation preserves everything

Prove the following. Use the results from 2(a) and 3 of last week's homework. You will have points taken off for proofs longer than 3 sentences.

(a) If g and g' are conjugate in G, they have the same order.
(b) If H and H' are conjugate subgroups in G, they have the same order.
(c) If H and H' are conjugate subgroups in G, they are isomorphic groups.

4. The Klein 4 group, a cappella

Recall from class that $\mathbb{Z}/2\mathbb{Z}$ is a cyclic group of order 2. Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. (This is also written $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ sometimes.) This has the "product" group structure you studied in the last homework. This example is called the *Klein four group.*

(a) How many elements are in G?
(b) Show that G is not cyclic.
(c) Explain why G is not isomorphic to $\mathbb{Z}/4\mathbb{Z}$.
(d) Find a subgroup of S_4 isomorphic to G. Write down the isomorphism explicitly. (Whenever you have to refer to an element of S_4, use cycle notation.)
(e) (*) Now go Google "Klein Four Group, Finite Simple Group (of Order Two)." How many math terms do you recognize?

5. Index 2 subgroups are normal

(a) Let G be a group. Show that any index 2 subgroup of G is a normal subgroup. (We will later see that a group may have order divisible by 2, but still not have an index 2 subgroup.)
(b) (*) More generally, suppose p is the smallest prime dividing $|G|$. If $H \subset G$ is a subgroup of index p, show it must be normal. (Hint: Examine the action of G on G/H. This problem will involve a few non-trivial steps.)

6. (*) Orbits and conjugation

(a) Let G act on a set X. Note that this defines an action of any subgroup H on X. Show that if H and H' are conjugate, then there exists a bijection ϕ between the set of orbits of the H-action, and the set of orbits of the H'-action.
(b) Using the bijection ϕ you construct, if two orbits are related by $O' = \phi(O)$, show that there is a bijection from the orbit O to the orbit O'.