Homework 4

1. Subgroups of $\mathbb{Z}$

In this problem, you will show that every subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some $n \geq 0$.

Let $H \subset \mathbb{Z}$ be a subgroup which contains some non-zero element. Let $n \in H$ be the least, positive integer inside H. Show that $H = n\mathbb{Z}$. (Hint: Remainders.)

2. Conjugation actions

The conjugation action of a group on itself is by far the most important group action in representation theory. A full understanding of the conjugation action can be illusive, and in many contexts, proves quite essential for research.

(a) Fix an element $g \in G$. Define a map $C_g : G \to G$ by $h \mapsto ghg^{-1}$. Show that C_g is a group isomorphism.

(b) Show that $C_g \circ C_{g'} = C_{gg'}$. In other words, the assignment $g \mapsto C_g$ defines a group homomorphism $G \to \mathrm{Aut}_{\mathrm{Group}}(G)$. So this defines *another* group action of G on itself. It is quite different from the action we have considered earlier, where all we had was a group homomorphism $G \to \mathrm{Aut}_{Set}(G)$. This new map, $G \to \mathrm{Aut}_{\mathrm{Group}}(G)$, is called the *conjugation action* of G on itself.

(c) If G is abelian, show that C_g is trivial for all $g \in G$.

3. Group isomorphisms in general

Since C_g is a group isomorphism from G to itself, it tells us a lot about the subgroups and elements of G. This is because of some general properties of group isomorphisms, which we now explore. Let $\phi : G \to H$ be a group isomorphism. If $K \subset G$ is a subset, we define

$$\phi(K) = \{h \in H \text{ such that } h = \phi(g) \text{ for some } g \in K\}.$$

(a) Show that isomorphisms preserve orders of elements. That is, show that if g is an element of order n, then $\phi(g)$ is.
(b) Show that if $K \subset G$ is a subgroup, it is isomorphic to $\phi(K)$.
(c) Show that isomorphisms preserve normal subgroups. That is, Show that if $K \subset G$ is a normal subgroup, then $\phi(K) \subset H$ is normal.
(d) Let K be a normal subgroup G. Show that there is a group isomorphism $G/K \cong H/\phi(K)$.

Throughout the following exercises, if you have time, think about what the above results imply about elements and subgroups of G that are conjugate.

4. Conjugacy classes of elements

(a) Two elements $g, g' \in G$ are called *conjugate* if there exists some $h \in G$ such that

$$h^{-1}gh = g'.$$

Show by example that if g and g' are conjugate, the choice of h need not be unique.

(b) Show that being conjugate defines an equivalence relation on the set G. That is, show that the relation "$g \sim g'$ if g is conjugate to g'" is an equivalence relation. Under this relation, the equivalence class of g is called the *conjugacy class* of g.

(c) Show that g is the only element in its conjugacy class if and only if g is in the center of G.

5. Conjugacy classes of subgroups

Let H and H' be subgroups of G. We say H and H' are *conjugate* if there is some g such that

$$C_g(H) = H'.$$

That is, if $gHg^{-1} = \{ghg^{-1}, h \in H\} = H'$ for some g.

(a) Show that being conjugate defines an equivalence relation on the set of all subgroups of G. That is, show that the relation "$H \sim H'$ if H is conjugate to H'" is an equivalence relation. The equivalence class of H under this relation is called the *conjugacy class of H*.

(b) Show that H is the only element in its conjugacy class if and only if H is normal.