Homework 3

1. Cosets of S_3 with respect to S_2

Let S_3 be the symmetric group on 3 elements. Recall that this is the set of all bijections from $\underline{3}$ to itself, where $\underline{3} = \{1, 2, 3\}$. Let $H \subset S_3$ be the set of all bijections $\tau : \underline{3} \to \underline{3}$ such that $\tau(3) = 3$—i.e., the subset of all bijections that fix 3.

(a) Show H is a subgroup of S_3.

(b) So H acts on $G = S_3$. How many elements are there in the orbit space? That is, how many orbits are there?

(c) Finally, write out each orbit explicitly. This means you must write out which elements of S_3 are in each orbit.

(d) For any $n \geq 1$, let $H \subset S_n$ be the subgroup of all elements that fix n. Exhibit an isomorphism from H to S_{n-1}.

(e) How many orbits are there of the action of H on S_n?

2. Cyclic groups

A group G is called *cyclic* if there exists $g \in G$ for which $\langle g \rangle = G$.

(a) Show that if two cyclic groups have the same order (finite or otherwise) then they must be isomorphic.

(b) Show that S_2 is cyclic.

(c) Show that $\mathbb{Z}$ is cyclic.

(d) Use Lagrange's theorem to show that any group of prime order must be cyclic. (Hint: Last homework.)

(e) Prove that for any integer $n \geq 1$, there exists a cyclic group of order n. For instance, as a subgroup of S_n, or of $GL_2(\mathbb{R})$, or of $\mathbb{C}^\times$.

3. Abelian groups

A group G is called *abelian* if for all $g_1, g_2 \in G$, we have $g_1 g_2 = g_2 g_1$.

(a) Show that S_n is not abelian for any $n \geq 3$.

(b) Show that any cyclic group is abelian. Conclude that S_n is not cyclic for any $n \geq 3$.

(c) Show that the center of an abelian group is the whole group.

4. Product groups

Let G and H be groups. Define a map

$$m : (G \times H) \times (G \times H) \to G \times H, \qquad m((g,h),(g',h')) = (gg', hh').$$

Note that throughout this problem, 1 may refer to either the group unit of G, or the group unit of H.

(a) Show that m defines a group structure on $G \times H$.

(b) Show that $(g,1) \cdot (1,h) = (1,h) \cdot (g,1)$.

(c) Show that if G and H are abelian, then $G \times H$ is abelian (with the above group structure).

(d) Show that the maps

$$G \to G \times H \qquad g \mapsto (g,1)$$

and

$$G \times H \to G, \qquad (g,h) \mapsto g$$

are group homomorphisms.