SECTION – WEEK 2 ON 9/9/2014

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These notes were written a few days after section was held, during which not much thought was given to what was said in section. As consequence, these notes will be rather short, for most of what I remember was just review of material, but I'll present a few examples and motivations that weren't mentioned in class. If any mistakes or imprecisions or confusions are found, please let me know!

1. Group Actions

Recall:

Definition 1.1. Let G be a group and X a set. Note $\mathrm{Aut}_{\mathsf{Set}}(X)$ forms a group under composition. A *group action* of G on X is a group homomorphism

$$G \xrightarrow{\phi} \mathrm{Aut}(X)$$

or alternatively, a map $\phi : G \to \mathrm{Aut}(X)$ such that:

- $\phi(1_G) = \mathrm{Id}_X$.
- $\phi(g_1 \cdot g_2) = \phi(g_1) \circ \phi(g_2)$.

The above definition really just says that we want a group to act on a set X in an invertible manner (since group elements are invertible!), and in a manner such that group multiplication corresponds to composition of automorphisms, so we can actually think of group elements as automorphisms on X themselves!

Remark 1.2. In class, we introduced the group action of a group G on itself, given by left translation. Namely, we have a map $G \xrightarrow{\phi} \mathrm{Aut}_{\mathsf{Set}}(G)$ such that $g_1 \overset{\phi}{\mapsto} \phi(g_1)$. Well, since $\phi(g_1)$ is a set-theoretic automorphism on G, I have to tell you what it does! Define $\phi(g_1) : h \mapsto g_1 \cdot h$. To understand what it's called "left translation", here's a concrete example:

Let $G = \mathbb{Z}$. Then given any $n \in \mathbb{Z}$, we have $\phi(n) : \mathbb{Z} \to \mathbb{Z}$ given by $\phi(n) : a \mapsto n + a$. So if $n > 0$, then this map $\phi(n)$ is literally pushing our number line from the left to the right, i.e. translating it!

Remark 1.3. Here's another example. Again, take G a group and $X = G$ as a set. Define

$$\phi' : G \to \mathrm{Aut}_{\mathsf{Set}}(G)$$

by $g \mapsto \phi'(g)$. You shouldn't be happy about this definition, since I just gave you this guy $\phi'(g)$ but didn't tell you what it was, namely how it acts on G as an automorphism of the set! Define $\phi'(g) : h \mapsto h \cdot g$, because why not? If we can left translate, why can't we right translate? Well, this is actually not correct...because it violates our axioms of a group action. I'll leave it as a very important exercise to you to see what goes wrong.[1]

So how do we fix this? First, I'll leave it to you to show that $(g \cdot h)^{-1} = h^{-1} \cdot g^{-1}$. And now, define $\phi(g) : h \mapsto h \cdot g^{-1}$. If you completed the above exercise, you'll see immediately why this works. This is called "right translation", and you can work with the example of $G = \mathbb{Z}$ for yourself as in *Remark* 1.2.

[1] Hint: It's the second bullet point in **Definition 1.1**. The idea is that this group operation doesn't have to commute, like in the matrix group $\mathrm{GL}_n(\mathbb{R})$.

Remark 1.4. We just got right and left translation as two example of group actions from the previous remarks, so why not compose them! I.e., define an action

$$G \xrightarrow{\phi} \mathrm{Aut}_{\mathsf{Set}}(G)$$

by $\phi(g) : h \mapsto g \cdot h \cdot g^{-1}$. For those of you who read Hiro's email, this should resemble the criterion for a subgroup to be normal. In fact, that's exactly part of it. This action is called conjugation by g, and it's very ubiquitous, from defining normal subgroups to studying group actions in many contexts (if you're interested in number theory like me, this kind of thing is used very often!) In fact, a subgroup $H \subset G$ is normal if and only if it's sent to itself under the conjugation action!

One more thing to say about conjugation is that it's not only a map of sets $G \to G$, it's a group homomorphism! I'm sure Hiro will talk about this in class, so I won't spoil it for you, but it's something that's very important to learn!

1.1. **Motivations for Group Actions.** As one of the greatest mathematicians has told me,[2] let the math speak for itself – don't feel like you need to motivate everything. If it's beautiful, it'll motivate itself. So, that said, I don't want to *have to* motivate group actions for you, but the history is actually quite interesting, so it's worth discussing :).

So, if you were a French or German mathematician in the mid-19th century, your definition of a group would not have been the one we gave you in class.[3] In fact, a group was simply defined to be the matrix group $\mathrm{GL}_n(\mathbb{R})$, over maybe $\mathbb{C}$ if you wanted to work in a world that makes everything absolutely beautiful.[4] Naturally, we had a bijection

$$\mathrm{GL}_n(\mathbb{R}) \simeq \mathrm{Aut}(V), \tag{1.1}$$

for V a real vector space of finite dimension n. Thus, it was natural to study how the elements of $\mathrm{GL}_n(\mathbb{R})$ behaved through the above bijection, instead of just row-reducing mindlessly – this "group action" in the automorphism (1.1) actually gives rise to the more abstract theory of linear algebra all of you learned before! This is actually something called a group representation, when you have a group homomorphism from G to the linear automorphism group of a vector space, which is something we'll also learn soon enough. The point is, group actions arose naturally from the study of the automorphism (1.1), so what we're studying isn't completely contrived. To see why/how we got from studying linear automorphisms of vector spaces to just automorphisms of sets, it turns out that this notion of group action is ubiquitous beyond the land of linear algebra, so why not "generalize" our theory of group actions from vector spaces to sets! It turns out that this type of procedure is very important in algebra. You want to see what you've got, try to take away any structure that you don't necessarily need to study the "abstract theory", and see if you have an interesting theory. It's this procedure that actually led Emmy Noether to define our current notion of group in the early 20th century.[5]

1.2. **What's to come.** [6]

From the construction of the free group, to just the definition of normal subgroup, it'd be nice if our group operations commuted, huh? I mean, every subgroup would be normal! And as we'll see, that's a very handy thing in many cases. Fortunately, in many important cases, our group operations do commute (e.g. think $G = \mathbb{Z}$ under addition or $G = \mathbb{C}^*$ under multiplication). As is in the PSet, this kind of group is known as an *abelian group*, named after the 18th century mathematician (I think 18th, at least), who developed the initial theory of groups with Galois. It turns out that studying abelian groups is exactly what a lot of professional mathematicians do. In

[2] I don't want to just put his/her name down for the sake of not just spreading things without prior consent...

[3] I promise you we're actually teaching the right stuff though.

[4] Think Fundamental Theorem of Algebra! Once you guys take 113, you'll also realize that analysis over $\mathbb{C}$ is absolutely beautiful too.

[5] And if any of you want to learn, there have been more "abstract" approaches to understanding groups, and I'd be more than happy to teach you if you ask!

[6] Sorry, Hiro!

fact, much of modern number theory is concerned with extending results about abelian groups to arbitrary groups, which is incredibly difficult, even with a huge number of people working on this (Langlands) program. We might also see how we can take certain "geometries"[7] and turn them into abelian groups! For example, the circle group S^1 is abelian, and that also tells us that the torus is abelian too![8] Don't take anything in this paragraph too seriously – I just want to help you anticipate arguably one of the most fundamental tools in every mathematician's life.

2. Miscellaneous

I've mentioned this to a few people already, but if you have any questions, feel free to email me questions or plan a time to meet for face-time (if I can find a time that I'm free...), or if I don't look too busy and you run into me, that's an appropriate time too!

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[7] Whatever this means... ;)

[8] The torus is $S^1 \times S^1$.