Yuan's talk Feb 17th

The action: $L = 2t \int dz^2 \left(\begin{array}{l} \frac{1}{2} g_{IJ}(\Phi)\, \partial_z \phi^I \partial_{\bar z} \phi^J + \frac{i}{2} g_{IJ} \psi_-^I D_z \psi_-^J \\ + \frac{i}{2} g_{IJ} \psi_+^I D_{\bar z} \psi_+^J + \frac{1}{4} R_{IJKL} \psi_+^I \psi_+^J \psi_-^K \psi_-^L \end{array} \right)$

[Witten: mirror mflds + TFTs]

Outline: 1) free boson, fermion [see Freed: 5 lectures on SUSY]

2) Lagrangian formulation of SUSY NLSM

1) free boson. Let $\phi: \mathbb{R} \longrightarrow \mathbb{R}$ $\quad S_B(\phi) = \int_{\mathbb{R}} |\nabla \phi|^2 = \int_{\mathbb{R}} \dot\phi^2(t)\, dt$

E-M on $\mathbb{R}^n$ = harmonic f^ns.

2) free fermion $\Psi: \mathbb{R}^{1|k} \longrightarrow \mathbb{R}$

$\Psi(t) = \psi^i(t)\,\theta_i$ with symp pairing

$$S_F(\Psi) = \frac{1}{2} \int_{\mathbb{R}} \Psi\dot\Psi - \dot\Psi\Psi\, dt = \int_{\mathbb{R}} \Psi\dot\Psi\, dt$$

eg. $k=1$: $S_F(\Psi) \equiv 0$ since $\theta \wedge \theta = 0$.

$k=2$: $S_F(\Psi) = \int_{\mathbb{R}} (\psi^1\theta_1 + \psi^2\theta_2)(\dot\psi^1\theta_1 + \dot\psi^2\theta_2)$

$= \int_{\mathbb{R}} (\psi^1\dot\psi^2 - \psi^2\dot\psi^1)\, \theta_1\theta_2\, dt.$

[secretly, $d\theta_1 d\theta_2$ present].

Feb 17th talk.

Standard susy NLSM

explain basic Lagrangian formulation

Σ = Riemann surface — coord $z, \bar{z}$

$X = (X, g)$ Riemannian mfld — coord. ϕ^I

$I = 1, \dots, n = \dim_{\mathbb{R}} X$

Consider $\Phi : \Sigma \longrightarrow X$ a smooth map, locally defined by

$\Phi^I = \Phi^I(z, \bar{z})$

All tangent, cotangent bundles etc will be complexified.

The metric g on X is an element of $\Gamma(Sym^2(T^*X))$

The example to keep in mind is $X = \cancel{\mathbb{C}}\ \mathbb{R}^2$. & g is a positive number (1 say).

in which case Φ is just a smooth f^n on Σ.

Consider $d\Phi : T\Sigma \longrightarrow TX$

or really $d\Phi \in \Gamma(\Sigma, T^*\Sigma \otimes \Phi^* TX)$

or really $\partial\Phi : T^{1,0}X \longrightarrow TX$

in co-ordinates, $\partial\Phi^I = \dfrac{\partial\Phi^I}{\partial z}(z, \bar{z})\, dz.$

similarly for $\bar{\partial}\Phi$. $\Rightarrow$ $\partial\Phi \wedge \bar{\partial}\Phi$ is a 2-form on Σ with values in $T_X^{\otimes 2}$.

$\Rightarrow$ Can define the Bosonic kinetic energy term as

$$\int_\Sigma \langle \partial\Phi \wedge \bar{\partial}\Phi \rangle_g$$

Remark: Actually for $E \in \mathrm{Tens}(T_X \oplus T^*_X)$

g defines the usual norm on sections of E.

So, $\Phi^* E \downarrow \Sigma$ has fibrewise norm

Example: why is this kinetic energy?

Σ = ellipre cure

In case $X = \mathbb{C}$, $g = 1$ (std). you just have

(up to a constant) $\Phi : \Sigma \longrightarrow \mathbb{C}$

$$\partial\Phi = \frac{\partial\Phi}{\partial z} dz \qquad \bar{\partial}\Phi = \frac{\partial\Phi}{\partial \bar{z}} d\bar{z}$$

$$\int_\Sigma \langle \partial\Phi \wedge \bar{\partial}\Phi \rangle_g = \int_\Sigma dz\, d\bar{z}\ \frac{\partial\Phi}{\partial z} \cdot \frac{\partial\Phi}{\partial \bar{z}}$$

integrate by parts & constant

$$= \int_\Sigma d\mathrm{vol}_\Sigma\ \Phi \cdot \underbrace{\frac{\partial^2}{\partial z \partial \bar{z}}}_{\Delta \text{ Laplacian}} \Phi$$

$$= \int_\Sigma |\nabla\Phi|^2 d\mathrm{vol}_\Sigma$$

[I think it is actually to write $\int_\Sigma |\nabla\Phi|^2 d\mathrm{vol}_\Sigma$ in general.]

("odd")

Lets proceed similarly with the fermionic terms.

No explanation of spinor bundles — wait for Jesse

Let K & $\bar{K}$ be canonical & anti-canonical bundles on Σ

$$\begin{array}{c} K \\ \downarrow \\ \Sigma \end{array} \quad \text{take a square root of both.}$$

$$\begin{array}{c} K^{1/2} \\ \downarrow \\ \Sigma \end{array} \quad \& \quad \begin{array}{c} \bar{K}^{\frac{1}{2}} \\ \downarrow \\ \Sigma \end{array}$$

Just like $\partial\Phi \in \Gamma(\Sigma, \overset{K_\Sigma}{\cancel{\Omega^{1,0}}} \otimes \Phi^* T_X)$

consider spinor bundles instead

$$\psi_+ \in \Gamma(\Sigma, K^{1/2} \otimes \Phi^* T_X)$$
$$\psi_- \in \Gamma(\Sigma, \bar{K}^{1/2} \otimes \Phi^* T_X)$$

(Fields are triples $(\underbrace{\Phi}_{\text{bosonic}}, \underbrace{\psi_+, \psi_-}_{\text{pair of fermions}})$.)

Now the fermionic term is given by

$$\int_\Sigma \langle \psi_+ \bar{D}_{\bar{z}} \psi_+ g \rangle$$ & analogous term for ψ_-

[remember $\int \psi \dot{\psi}$?]

$\bar{D}$ is an (partially) antiholomorphic connection $\bar{D} : K^{1/2} \otimes \Phi^* T_X \longrightarrow (\text{same}) \otimes \bar{K}$

$(K^{1/2})^{\otimes 2} \otimes \bar{K}$ is a top form on Σ & can be integrated

What is the $\bar{D}$ operator?

$$\bar\partial\text{-connection} \quad K^{1/2} \otimes \Phi^* T_X \quad \text{pullback of LC connection}$$

$$\downarrow$$

$$\Sigma$$

$$\bar{D} = \bar\partial \otimes \mathrm{Id} \pm \mathrm{Id} \otimes \Phi^* \nabla^{LC}$$

In a local hol$^{\underline{c}}$ trivialization of $K^{1/2}$, i.e. just consider

$\Phi^* T_X \to \Sigma$

recall that $\Phi_* : T_\Sigma \to T_X$ s.t.

$$\Phi_* \left(\frac{\partial}{\partial \bar z}\right) = \sum_J \frac{\partial \phi^J}{\partial \bar z} \frac{\partial}{\partial \Phi^J}$$

you might call ∂_J on X

. Let Γ_{IJ}^{K} be defined as usual by

$$\nabla^{LC}_{\partial_I} \partial_J = \Gamma_{IJ}^{K} \partial_K$$

Then for $\psi_+ = \psi_+^I \frac{\partial}{\partial \Phi^I}$, use Leibnitz rule $\nabla(fX) = df + f\nabla X$

$$\bar{D}_{\frac{\partial}{\partial \bar z}} \psi_+ = \underbrace{\frac{\partial}{\partial \bar z}\psi_+^I(z,\bar z)\, \partial_I}_{\text{pulled back}} + \underbrace{\psi_+^I \frac{\partial \phi^J}{\partial \bar z} \underbrace{\nabla^{LC}_{\partial_J} \partial_I}_{\Gamma_{JI}^{K} \partial_K}}_{\frac{\partial \phi^J}{\partial \bar z} \Gamma_{JI}^{K} \psi_+^I \partial_K}$$

reindex $K \leftrightarrow I$

$$= \frac{\partial \phi^J}{\partial \bar z} \Gamma_{JK}^{I} \psi_+^K \partial_I$$

so more in co-ord,

$$\bar{D}_{\bar z} \psi_+^I = \frac{\partial \psi_+^I}{\partial \bar z} + \frac{\partial \phi^J}{\partial \bar z} \Gamma_{JK}^{I} \psi_+^K$$

Finally, the interaction term

This is fairly simple. Recall Curvature is tensorial.

$$R_{IJKL} = \langle (\nabla_I \nabla_J - \nabla_J \nabla_I) \partial_K, \partial_L \rangle_g$$

and $\psi_+ = \psi_+^I \partial_I$ [a bit confused here]

Recall that usually $\langle R(X,Y) Z, W \rangle$

satisfies $R(X,Y) = -R(Y,X)$

But here ψ's anti-commute

$$\Rightarrow \int_\Sigma \langle R(\psi^+, \psi^+)\psi^-, \psi^- \rangle \quad \text{makes sense}$$

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