Twistor theory references
Mason
Ward + Wells
+ web.

Twistor Constructions of SUSY gauge theories

ref for SUSY on Twistor.
B? Mason - Skinner

Notation: $V_{\mathbb{R}} = \mathbb{R}^4$ translation $\quad V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$

$Spin(4) = SU(2)_+ \times SU(2)_-$ $\quad$ & $V_{\mathbb{C}} \overset{\rho}{\xleftarrow{\cong}} S^+ \otimes S^-$

$S^+ \quad S^-$ defining representations

Choose W a complex v. space. Then $T^W = V_{\mathbb{C}} \oplus \Pi(S^+ \otimes W \oplus S^- \otimes W^-)$

$\alpha^+ \otimes w \in S^+\otimes W \qquad \alpha^- \otimes w' \in S^-\otimes W^-$

$[\alpha^+ w, \alpha^- w'] = \rho(\alpha^+ \otimes \alpha^-) \langle w, w' \rangle.$ $\quad$ Spin 4 acts everywhere

A SUSY field theory is one where the Lie alg. of translations are extended to T^W.

Twistor Space

def$^=$ $\mathbb{PT}$ is the total space of the bundle $\mathcal{O}(1) \otimes S^-$ over $\mathbb{P}(S^+) \cong \mathbb{P}^1$

$SU(2)_-$ acts on S^-; $SU(2)_+$ lifts to $\mathcal{O}(1)$

& is 3 $\mathbb{C}$-dimensional. $\quad$ spin4 acts!

$\mathbb{PT} \cong \mathbb{P}^3 \setminus \mathbb{P}^1$ — basic version.

$V_{\mathbb{C}}$ also acts. $\quad \mathbb{PT} \xrightarrow{pr} \mathbb{P}(S^+)$

relative tangent bundle $T_{pr} = \pi^* \mathcal{O}(1) \otimes S^-$

$\Rightarrow$ have map $\underbrace{H^0(\mathbb{P}(S^+), \mathcal{O}(1) \otimes S^-)}_{} \longrightarrow Vect(\mathbb{PT})$

$\quad \parallel$ $\quad$ global sections

$\underbrace{H^0(\mathbb{P}(S^+), \mathcal{O}(1))}_{(S^+)^\vee \cong S^+} \otimes S^-$

Concretely, $x \in V_{\mathbb{C}}$ gives a section of $\mathbb{PT} = \mathcal{O}(1) \otimes S^- \longrightarrow \mathbb{P}^1$

Call the image $\sigma_x \subset \mathbb{PT}$

Lemma: $\forall p \in \mathbb{PT}\ \ \exists!\ x \in V_{\mathbb{R}} = \mathbb{R}^4$ with $p \in \sigma_x$

$\Rightarrow$ get a map $\mathbb{PT} \longrightarrow V_{\mathbb{R}} = \mathbb{R}^4$ a non-holomorphic fibration.

fibre at x is σ_x.

$\mathbb{C}$ lines in $\mathbb{H}^2$

or $\mathbb{P}^3_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{H}} = S^4$ $\mathbb{H}$ lines in $\mathbb{H}^2$

take $\mathbb{C}$-line $\longrightarrow$ $\mathbb{H}$ line it spans

Why! Penrose-Ward correspondence

Thm There is a natural bijection (equivalence of categories) between

1) complex ASD bundles on $\mathbb{R}^4$

anti-self-dual (any group) G bundle for G = cplx group, ASD connection

& 2) hol^m bundles on $\mathbb{PT}$ which are trivial on all twistor fibres σ_x.

Example: If $U \overset{\text{open}}{\subseteq} \mathbb{R}^4$ then $\{$harmonic f$^{\underline{n}}$s on $U\}$

$$= H^1_{\bar\partial}\left(p^{-1}(U), \mathcal{O}(-2)\right)$$

Before seeing SUSY gauge theory can we see non-SUSY field theory?

ex. self-dual YM. $A \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$ connection

$B \in \Omega^2_+(\mathbb{R}^4, \mathfrak{g})$ selfdual 2-form ad $\mathfrak{g}$ valued

$S(A,B) = \int F(A)_+ \wedge B$ with usual gauge symmetry ex. Hol BF theory on $\mathbb{PT}$

fields are: $\alpha \in \Omega^{0,1}(\mathbb{PT}, \mathfrak{g})$ $\bar\partial$ operator

$\beta \in \Omega^{3,1}(\mathbb{PT}, \mathfrak{g})$

& ghosts etc

$\Omega^{0,*}\mathfrak{g}[1]$

$\Omega^{3,*}\mathfrak{g}[1]$

$F^{0,2}(\alpha)$ is obstruction to α being an integrable $\bar\partial$-operator

$S(\alpha,\beta) = \int F^{0,2}(\alpha) \wedge \beta.$

self-dual YM deforms to ordinary YM by adding $c \cdot \int B \wedge B$ to the action.

self-dual limit is $c = 0$. used for $\mathbb{PT}$

SDYM is equivalent to hol^m BF theory on $\mathbb{PT}$

SDYM given by $T^*[-1]$ (ASD bundles)

& hol BF —— $T^*[-1]$ (hol^m bundles on $\mathbb{PT}$)

these bases same!

To SUSY. For super-twistor space, a similar story.

Recall $T^W = V_{\mathbb{C}} \oplus \Pi(S^+ \otimes W \oplus S^- \otimes W^*)$ (no point in taking Dolbeault resolution of odd directions).

θ_i —— w_i odd

lin coord.

$\mathbb{P}\mathbb{T}^W$ = total space of $\mathcal{O}(1) \otimes (S^- \oplus \Pi W)$

over $\mathbb{P}(S^+)$

$\{w_i \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial w_i}\}$ gives translation

$\mathcal{O}_{\mathbb{P}\mathbb{T}^W} = \Lambda^*(\mathcal{O}(-1) \otimes W^*)$ on $\mathbb{PT}$

globally, $\Omega^{0,*}(\mathbb{PT}, \mathcal{O}_{\mathbb{P}\mathbb{T}^W})$

$S^- \otimes W^* \ni \alpha$ gives a bundle map $W \otimes \mathcal{O}(1) \to S^- \otimes \mathcal{O}(1)$

other half (rotated v. fields) $H^0(\mathbb{P}(S^+), \mathcal{O}(1) \otimes W) = S^+ \otimes W \subseteq$ odd Vect $(\mathbb{P}\mathbb{T}^W)$

vector fields along the fibres $\mathbb{PT} \to \mathbb{P}^1$

So what is SUSY YM (self-dual limit) — at particular value of coupling constant

$N=1, N=2$ (dim W)

$T^*[-1]$ (hol bundles on $\mathbb{P}\Pi^W$, trivial on every twistor fiber).

In the abelian case, $G = \mathbb{C}$

$N=1$ deformations of a bundle are $\Omega^{0,*}(\mathcal{O}_{\mathbb{P}\Pi^W})$ — in general some exterior algebra

$$= \underbrace{\Omega^{0,*}\left(\mathcal{O}_{\mathbb{P}\Pi} \oplus \Pi\mathcal{O}_{\mathbb{P}\Pi}(-1)\right)[1]}$$ — H^1 is target space

dual. $\Omega^{3,*}\left(\mathcal{O}_{\mathbb{P}\Pi} \oplus \Pi\mathcal{O}_{\mathbb{P}\Pi}(1)\right)[1]$.

Using that $K_{\mathbb{P}^3} = \mathcal{O}(-4)$, this all becomes

$$\Omega^{0,*}\begin{pmatrix} \overset{-1}{\mathcal{O}} & \overset{0}{\mathcal{O}(-1)} \\ \mathcal{O}(-4) & \mathcal{O}(-3) \end{pmatrix}$$

$\underbrace{\mathcal{O},\ \mathcal{O}(-4)}_{\text{ordinary } N=0 \text{ YM}}$ $\underbrace{\mathcal{O}(-1),\ \mathcal{O}(-3)}_{\text{spinor \& antifield}}$

$N=2$: $\Lambda^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$

get

$$\Omega^{0,*}\begin{pmatrix} \Lambda^*(\mathcal{O}(-1)^{\oplus 2}) + \text{dual, i.e.} & & \\ \text{odd} & \text{even} & \text{odd} \\ \mathcal{O} & \mathcal{O}(-1)^2 & \mathcal{O}(-2) \\ \mathcal{O}(-4) & \mathcal{O}(-3)^2 & \mathcal{O}(-2) \end{pmatrix}$$

A susy operator induces differential (from S^+ say) $\mathcal{O}(-1) \to \mathcal{O}$ ← fields.

2 massless scalars (harmonic f's)

$H^1(\mathcal{O}(-2))$ corresponds to harmonic f's.

What is the action?

$N=1$: Fields $\alpha \in \Omega^{0,*}(\mathfrak{g} \oplus \Pi\mathcal{O}(-1) \otimes \mathfrak{g})[1]$

$\oplus\, \Omega^{3,*}(\mathfrak{g} \oplus \Pi\mathcal{O}(1) \otimes \mathfrak{g})[1]$

$$S(\alpha, \beta) = \int \langle \bar\partial \alpha, \beta \rangle + \tfrac{1}{2} \langle [\alpha, \alpha], \beta \rangle$$

$N=4$: different

$\mathbb{P}\mathbb{T}^{N=4}$ = ~~total~~ space of $\mathcal{O}(1) \otimes S_- \oplus \Pi(\mathcal{O}(1)^4)$

observe $= \left(\mathbb{C}^{4|4} - 0\right) / \mathbb{C}^*$ (projective space of a supermfld)

$c_1(\mathbb{P}\mathbb{T}^{N=4}) = 0$, & so $K_{\mathbb{P}\mathbb{T}^N}$ is trivial.

$\mathcal{O}_{\mathbb{P}\mathbb{T}^{N=4}} = \Lambda^* \mathcal{O}(-1) \longrightarrow K_{\mathbb{P}\mathbb{T}} = \mathcal{O}(-4)$

$\Rightarrow$ we can integrate sections of $\mathcal{O}_{\mathbb{P}\mathbb{T}^{N=4}}$. A <u>super CY</u>

deformation as principle G-bundle

On any CY have hCS. $\alpha \in \Omega^{0,*}(\mathbb{P}\mathbb{T}, \mathcal{O}_{\mathbb{P}\mathbb{T}^{N=4}} \otimes \mathfrak{g})[1]$

& $S(\alpha) = \int \frac{1}{2} \langle \alpha \wedge \bar\partial \alpha \rangle + \frac{1}{6} \langle \alpha, [\alpha, \alpha] \rangle$.

have vol form.

EoM = moduli of bundles on $\mathbb{P}\mathbb{T}^{N=4}$.