Jesse's talk.

Superspace: $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$

$(f, \phi) \quad \phi : \mathcal{O}_X \to f^* \mathcal{O}_Y$

$\mathbb{R}^{p,q} = (\mathbb{R}^p, C^\infty_{\mathbb{R}^p}[\theta_1, \dots \theta_q])$ superspace locally.

(M, Ω^k_M) dim $n|n$

super comm. (classical)

def quant $\longrightarrow$ Dirac operators.

comm $\longrightarrow$ ass.

Laplacians. on $\mathbb{R}^2$, std: $\Delta = \Sigma \partial_i^2$

Heat eq^n $\partial_t - \Delta = 0$

Wave eq^n $\partial_t^2 - \Delta = 0$.

Laplace - Beltrami.

$\Delta_g = [d, d^*] = g^{ij} \partial_i \partial_j + \text{l.o.t.}$

ex 1d Wave $(\partial_t - \partial_x)(\partial_t + \partial_x) u = 0 \quad u(0, x) = \phi(x)$

$u(t, x) =$ sum of left + right moving waves

$= \frac{1}{2}[\phi(x+t) + \phi(x-t)]$.

governs relativistic Electromagnetism.

Clifford algebras + Dirac operators

$\not{D}$ is 1st order odd diff^l operator on a superbundle s.t $\not{D}^2 = \Delta_{\text{generalized}}$.

Picking g defines a canonical deformation of (M, Ω^*_M) to a non-comm. supermfld (M, Cl_g)

$Cl_g = \text{Tens}(\Omega^1_M) / [\omega, \omega'] = -2g(\omega, \omega')$.

View as def quant.

$A^t_g = Cl_{tg}$. $\quad t \to 0$ obtains Ω^*_M.

$t \to 1 \quad Cl_g$.

algebra different - same v. bundle.

Index theory = Study diff$^{\underline{l}}$ operators on (M, Cl_g).

A Clifford module $\mathcal{E}$ is a super v. bundle on M w/ action

$$Cl_g \otimes \mathcal{E} \longrightarrow \mathcal{E}$$

Canonical example. $Cl_g \otimes \Omega^*_M \longrightarrow \Omega^*_M$

$$\gamma \cdot \alpha \overset{\text{def}}{=} \underbrace{\gamma \wedge \alpha}_{\varepsilon(\omega)\alpha} - \alpha(\gamma^\flat)$$

$\iota_{\gamma^\flat}$ change form to v. field.

∇^{LC} admits unique lift to Cl_g

A Clifford connection on module $\mathcal{E}$ satisfies: $\nabla^{\mathcal{E}}(\gamma\alpha) = \nabla^{LC}(\gamma)\cdot\alpha + (-1)^{|\gamma|}\gamma\cdot\nabla^{\mathcal{E}}_\alpha$

$\nabla^{\Omega^*_M}$ is a Clifford connection for the Cl_g-structure we defined.

<u>Def$^{\underline{n}}$</u>. The Dirac operator assoc to (M, g), $(\mathcal{E}, \nabla^{\mathcal{E}})$ is:

$$\not{D}_{\mathcal{E}} : \mathcal{E} \xrightarrow{\nabla^{\mathcal{E}}} \Omega^1_M \otimes \mathcal{E} \xrightarrow{\text{Clif}} \mathcal{E}$$

example case $(\Omega^*_M, \nabla^{\Omega^*_M})$ has Dirac operator $d + d^*$.

Want: Clifford modules for which Clifford-action is skew-adjoint.

If $\nabla^{\mathcal{E}}$ is Clifford connection compatible w/ metric, then $\not{D}_{\mathcal{E}}$ is self-adjoint.

$$\not{D}_{\mathcal{E}} = \begin{pmatrix} 0 & \not{D}^+_{\mathcal{E}} \\ \not{D}^-_{\mathcal{E}} & 0 \end{pmatrix} \qquad \not{D}^-_{\mathcal{E}} = (\not{D}^+_{\mathcal{E}})^*$$

Observe : 1) $\not{D}_\varepsilon$ is odd 1^{st} order diff^{l} operator

2) $\not{D}^2$ factorizes $(\partial_t^2 - \Delta) = (\partial_t - \not{D})(\partial_t + \not{D})$

3) Complexify – bring in cplx conj.
If g metric Δ_g on $\Omega_M^\bullet \otimes \mathbb{C}$ self adjoint
then its evals are real & non-neg.

Suppose $\not{D}$ self-adjoint. For non-0 eigenstate u, $\Delta u = \lambda u$, $\not{D}u$ is also
Δ- homogeneous
λ- eval, & non-zero.

$$\Delta(\not{D}u) = \not{D}(\Delta u) = \lambda \not{D}u.$$

gives iso $\mathcal{E}^+_{\neq 0} \cong \mathcal{E}^-_{\neq 0}$

$\mathbb{Z}/2$-grading

$\Rightarrow \mathcal{E}^+_\lambda \cong \mathcal{E}^-_\lambda$ for $\lambda > 0$. & $\mathrm{ind}(\not{D}) = \dim \mathcal{E}^+_0 - \dim \mathcal{E}^-_0$

On a compact mfld, the spectrum of Δ is discrete, e-spaces finite dim'l
this part needs Riemannian signature.

$\mathrm{Ind}(\not{D})$ independent of metric & connection,
a topological invariant.

the SUSY is that if for u_λ $\lambda \to 0$ $\mathcal{E}^+_0$; $\not{D}u_\lambda$ super partner $\mathcal{E}^-_0$ & vice-versa.

Spinors

reduce to $SO(n)$.

Let $Cl(n) = \mathbb{R}[\varepsilon_1, \dots, \varepsilon_n] / \varepsilon_i^2 = -1$
$\varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 0$ for $i \neq j$

$\exists$ v.s. $S_{\mathbb{C}}$ s.t. $Cl_{\mathbb{C}}(n) \cong End(S)$.

$\exists!$ irrep of $so(n)$ that does not descend to a rep of $SO(n)$. $\widetilde{SO(n)} = Spin(n)$
(not s.c).

$Spin(n) \subseteq Cl^0(n)$ even piece

If we can lift the structure group of M to $Spin(n)$, ie
if $\exists$ $P_{spin}(M) \longrightarrow M$ w/
$P_{spin(n)}(M) \times^{Spin(n)} \mathbb{R}^n \cong TM$, then we can form spinor bundle of M as $P_{spin(n)}(M) \times^{Spin(n)} S_{\mathbb{C}}$

We have a metric and Clifford connection on S_M & this defines $\not{D}_S$ as before.

<u>K-theory</u>

$K^0(M) = \{E^+ \oplus E^- \longrightarrow M\}/\sim$ If $E^+ \cong E^-$
then $E^+ \oplus E^- \sim 0$.

In $H^{\cdot}_{dR}(-)$ an orientation & metric allows us to define $dvol_g$
Poincaré-duality $\int_M -$

In K-theory. K-homology classifies elliptic operators over mfld
$\not{D}_S$ is the orientation class
$(S_{\mathbb{C}}, \nabla^{LC})$ is the volume form

K-theory family index thm.

fibration $M \longrightarrow B$ s.t. the relative tangent bundle $TV(M)$ is spin (vertical)

then we have $\int_F : K^*(M) \longrightarrow K^*(B)$

$$(E, \nabla^E) \longmapsto \operatorname{ind}(\not{D}_{S_F}, E)$$

$$\begin{array}{ccc} K^*(M) & \xrightarrow{ch(-)} & H^*_{dR}(M) \\ {\scriptstyle \operatorname{ind}(\not{D}_F, -)} \downarrow & & \downarrow {\scriptstyle \int_F \hat{A}(F) \wedge (-)} \\ K^*(B) & \xrightarrow[ch(-)]{} & H^*_{dR}(B) \end{array}$$

(Td class for spin-$\mathbb{C}$)

If B is a point, $\operatorname{ind}(\not{D}_M) = \int_M \hat{A}(M)$

$$\operatorname{ind}(\not{D}_E) = \int_M \hat{A}(M)\, ch(E).$$