Note Title 2/20/2012

Supermflds in Index Theory

(D) A superspace is a locally ringed space $(X, \mathcal{O}_X)$ s.t. $\mathcal{O}_X$ is a sheaf of superalgebras.

Morphisms are $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ as usual for ringed spaces.

Example: $\mathbb{R}^{p|q} = (\mathbb{R}^p, \underbrace{C^\infty(\mathbb{R}^p)}_{\substack{\text{even} \\ \text{generators}}}[\underbrace{\theta_1, \ldots, \theta_q}_{\substack{\text{odd} \\ \text{generators}}}])$

(D) A supermfld of dimension $p|q$ is a superspace locally isomorphic to $\mathbb{R}^{p|q}$.

Example: (M, Ω^*_M), M - smooth mfld

$$\Omega^*_M = \Omega^{ev}_M \oplus \Omega^{odd}_M$$

Ω^*_M is a sheaf of commutative superalgebras.

Recall supercommutator $[a, b] = ab - (-1)^{ab} ba$.

Laplacians

Recall ordinary Laplacian on $\mathbb{R}^2$:

Δ is the second order differential operator

$\Delta = \partial_x^2 + \partial_y^2$

Examples: Heat operator: $\partial_t - \Delta$

Wave operator: $\partial_t^2 - \Delta$

Globally: (M^n, g) an oriented Riemannian mfld.

g defines Hodge $*$ on Ω^*_M as follows:

volume form $dvol_g = \sqrt{\det g^{-1}}\, dx_1 \wedge \dots \wedge dx_n$ in coords

defines a non-degenerate pairing $\Omega^k(M) \otimes \Omega^{n-k}(M) \longrightarrow \mathbb{R}$

$(\alpha, \beta)_g := \int_M \alpha \wedge \beta$, so $\Omega^k(M)^\vee \cong \Omega^{n-k}(M)$.

Then $*\alpha \in \Omega^{n-k}(M)$ is the dual of α under this pairing.

e.g. on $\mathbb{R}^2$ w/ standard metric,

$$*dx_1 = dx_2$$

$$*dx_2 = -dx_1$$

Similarly, have adjoint to exterior derivative d^*
Call $d^*: \Omega^*_M \to \Omega^{*-1}_M$ the "divergence operator."

d & d^* are both odd operators on Ω^*_M.

$\Rightarrow [d, d^*] = dd^* + d^*d$.

(D) The Laplacian on (M,g) is the second order even differential operator $\Delta_g := [d, d^*] : \Omega^*_M \to \Omega^*_M$

Example: In coords $\Delta_g = g^{ij}\partial_i\partial_j$ + l.o.t.

This allows us to define heat & wave eqns on (M,g) as on $\mathbb{R}^n$.

History: 1D wave operator: $\partial_t^2 - \partial_x^2$

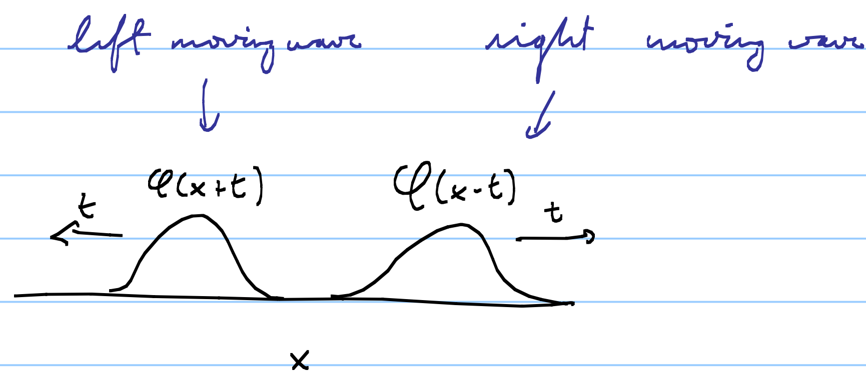
Can factor this as $(\partial_t - \partial_x)(\partial_t + \partial_x)$

Two first order operators and we can just read off soln:

$$(\partial_t - \partial_x)(\partial_t + \partial_x) u = 0$$

$$u(0,x) = \varphi(x)$$

$$\Rightarrow \quad u(t,x) = \tfrac{1}{2}\left(\varphi(x+t) + \varphi(x-t)\right)$$



Dirac, thinking about role of wave eqn in electromagnetism wanted a similar factorization in higher dimensions.

Modern formalism of Clifford algebras & Dirac operators gives this factorization:

(M, g) as above. g defines a canonical deformation of Ω^*_M to a sheaf of non-commutative superalgebras $Cl_{M,g}$

$Cl_{M,g}$ is the global sections of the bundle of Clifford algebras of (M,g).

(D) $$Cl_{M,g} := T^{\otimes}(\Omega^1_M) \Big/ \left([\omega_1, \omega_2] = -2(\omega_1, \omega_2)_{g^*} \right)$$

metric on forms

$\otimes$ over C^∞_M — degree 1 generators — graded commutator in $T^{\otimes}$

Note, $$\Omega^*_M = T^{\otimes}(\Omega^1_M) \Big/ \left([\omega_1, \omega_2] = 0 \right)$$

$\Rightarrow$ We have a 1-parameter family of sheaves of superalgebras

$$A^t_{M,g} := T^{\otimes}(\Omega^1_M) \Big/ \left([\omega_1, \omega_2] = -t(\omega_1, \omega_2)_{g^*} \right)$$

$$A^0_{M,g} = \Omega^*_M, \quad A^2_{M,g} = Cl_{M,g}.$$

e.g. $\mathbb{R}^2$ w/ standard metric

$$Cl_{\mathbb{R}^2} = C^\infty(\mathbb{R}^2)[dx_1, dx_2] \Big/ \begin{array}{l} dx_i^2 = -1 \\ dx_1 dx_2 = -dx_2 dx_1 \end{array}$$

(R) As vector bundles, $Cl_{M,g} \cong \Omega^*(M)$

(i.e. $P_{SO(n)}(M) \times^{SO(n)} Cl(\mathbb{R}^n) \cong \Lambda^* T^* M$)

For index theory, we study the supermfld

$$(M, Cl_{M,g}).$$

(D) A Clifford module is a locally free sheaf of super-modules for $Cl_{M,g}$.

Example: Recall $(-)^\flat : \Omega^1_M \to \mathrm{Vect}(M)$ given by $\omega^\flat \lrcorner \alpha = (\alpha, \omega)_{g^*}$.

$$Cl_{M,g} \otimes \Omega^*_M \longrightarrow \Omega^*_M \quad \text{given by}$$

$$\omega \cdot \alpha = \omega \wedge \alpha - \iota_{\omega^\flat}(\alpha)$$ gives Ω^*_M the structure of a Clifford module.

The Levi-Civita connection ∇^g lifts to a ! connection on $Cl_{M,g}$.

(D) a Clifford connection $\nabla^{\mathcal{E}}$ on a Clifford module $\mathcal{E}$ is a connection s.t. for $c \in Cl_{M,g}$, $\alpha \in \mathcal{E}$

$$\nabla^{\mathcal{E}}(c\alpha) = \nabla^g(c)\alpha + (-1)^{|c|} c \nabla^{\mathcal{E}} \alpha$$

Example: The Levi-Civita connection lifts to a connection on Ω^*_M.

This is a Clifford connection for the canonical Clifford action on Ω^*_M.

(D) Let $(\mathcal{E}, \nabla)$ be a Clifford module w/ connection. Then the associated Dirac operator is

$$\not{D}_{\mathcal{E}} : \mathcal{E} \xrightarrow{\nabla} \Omega^1_M \otimes \mathcal{E} \xrightarrow{c} \mathcal{E}$$

Example / Exercise:
The Dirac operator on $(\Omega^*_M, \nabla^{LC})$ is $d + d^*$.

Note: $(d+d^*)^2 = dd^* + d^*d = \Delta_g$

(R) Given a Clifford module w/ metric & metric connection, we can define a generalized Laplacian on $\mathcal{E}$.

We could have defined a Dirac operator as any first order differential operator whose square is a Laplacian. All Dirac operators arise via a Clifford connection as above.

Key properties of Dirac operators:

1) $D\!\!\!/_{\mathcal{E}}$ is an odd, first order differential operator; write $D\!\!\!/_{\mathcal{E}} = \begin{pmatrix} & D\!\!\!/^{+} \\ D\!\!\!/^{-} & \end{pmatrix}$

2) If $(\mathcal{E}, \nabla, \langle , \rangle_{\mathcal{E}})$ is a metric Clifford module s.t. Clifford action is skew-adjoint & ∇ is metric, then $D\!\!\!/_{\mathcal{E}}$ is skew-adjoint.

3) Let u be a homog. eigenstate for $\Delta_{\mathcal{E}}$, i.e. $\Delta_{\mathcal{E}} u = \lambda u$. (Assume all bundles are complex).

$\Delta_{\mathcal{E}}$ is self-adjoint $\Rightarrow$ $\lambda \in \mathbb{R}$.

In many cases of interest, $\Delta_{\mathcal{E}}$ is positive, i.e. $\lambda \in \mathbb{R}_{\geq 0}$.

Then if $\lambda \neq 0$, we have $\lambda u = \Delta_\varepsilon u = (\not{D}_\varepsilon)^2 u$

$\Rightarrow$ $\lambda^{-1} \not{D}_\varepsilon u \in \mathcal{E}^{-u}$ gives $\not{D}_\varepsilon(\lambda^{-1} \not{D}_\varepsilon u) = u$

& $(\not{D}_\varepsilon^2)(\lambda^{-1} \not{D}_\varepsilon u) = (\not{D}_\varepsilon)(u)$

$$= \lambda(\lambda^{-1} \not{D}_\varepsilon u)$$

i.e. $\not{D}_\varepsilon$ induces a bijection between the odd & even degree components of the positive eigenspaces of Δ_ε.

$$u \longleftrightarrow \lambda^{-1} \not{D}_\varepsilon u$$

D) We define the index $\operatorname{ind}(\not{D}_\varepsilon)$ of a self-adjoint Dirac operator to be the difference

$$\operatorname{ind}(\not{D}_\varepsilon) = \dim \ker \not{D}_\varepsilon^+ - \dim \ker \not{D}_\varepsilon^-.$$

Want to cover: • topological invariance of $\operatorname{ind}(\not{D}_\varepsilon)$ (invariance of metric)

- K-theory picture of index theorem
- index formula
- heat kernel approach to local & families index.