SUPERNOTES ON SUPER YANG-MILLS FOR SUPERSTUDENTS

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ABSTRACT. There are notes from a talk given in March of 2012 at the Simons Center Workshop on Supersymmetric Field Theories and Their Implications.

CONTENTS

Acknowledgments	1
Context	1
1. Why supersymmetry?	1
2. Superspace for $N = 1$	3
2.1. Superfields for $N = 1$ and $V = \mathbb{R}^{1,3}$	3
2.2. Superalgebra action	4
3. Supersymmetric actions and chiral fields	5
3.1. One verification of SUSY invariance	5
4. An aside about $N = 2$ SYM	8
5. N=1 Super Yang Mills	8
5.1. Pure Yang-Mills Action	8

Acknowledgments. First off, I learned all of this from Si Li, so thank you to Si for teaching me everything. I'm sure there are some over-simplifications and errors in my notes and talk, which are due only to me. Thank you also to the organizers (Dan Freed, Constantin Teleman, and Greg Moore) and the student organizers (Chris Elliott in particular) for giving me the chance to talk about this material.

Context. Also, these notes are for a talk given on Wednesday, in a workshop which began on Monday. As such I assume prior knowledge of things like the super-space construction and multiplets, so one should not take this as an introduction to supersymmetry in general.

1. WHY SUPERSYMMETRY?

1.0.1. *Killing infinities.* There's a lot of physical reasons to learn about supersymmetry, as we learned this week, but there are a lot of mathematical reasons as well. For instance, the degeneracies and singularities that show up in some field theories can be killed off by introducing supersymmetries. In super Yang-Mills, degeneracies only occur in the one-loop expansions, for instance.

1.0.2. *Localization to finite dimensions.* Further, SUSY models can be computed. When we have a path integral

$$\int D\phi e^{S[\phi]}$$

over some infinite-dimensional space, the path integral for SUSY theories can often be localized to a finite-dimensional space. Greg Moore talks about this in his notes from this workshop. I've also had conversations with Eric Zaslow where he also motivated super-symmetry as a method for reducing things to finite dimensional calculations. This is probably the most important relevance to mathematics. For instance, the space of *holomorphic* maps into a complex manifold is finite-dimensional for a fixed Riemann surface. This shows up in the A model, where we discover Gromov-Witten theory.

1.0.3. *The utility of Q.* Another huge plus is the importance of the super-symmetry operators Q. There's also been talk this week of why we can set a parameter t to go to ∞ or to 0, and one simple explanation for this is that Q-exactness buys you a lot. Often times we have a action which looks like

$$\int D\phi \exp(Q\text{-exact term} + \text{topological } Q\text{-closed term}).$$

We write the Q-exact term as $Q(A)$ for some expression A. If we scale the Q-exact term by a parameter t, we see that there is no dependence on t:

$$\begin{aligned} \frac{d}{dt}\int D\phi e^{tQ(A)} &= \int D\phi (QA) e^{tQ(A)} \\ &= \int D\phi Q(Ae^{tQ(A)}) \\ &= 0. \end{aligned} \tag{1}$$

(We have used the fact that $Q^2 = 0$.) In general if we want to compute the expectation value of B, if B is Q-exact, then $\langle B \rangle = 0$. In other words, the correlation function is only sensitive to Q-cohomology.

There are many examples in which this kind of limit is useful. For instance, if the action is $t|\overline{\partial}\phi|^2$, taking $t \to \infty$, we see the action must localize to where $\overline{\partial}\phi = 0$.

The same computation also occurred in the example of Morse theory we saw earlier this week, and the passage from Seiberg-Witten theory to Donaldson theory also goes through a bridge like this.

1.0.4. *Why $N = 2$ Super Yang-Mills?* The $N = 2$ Super Yang-Mills theory packages together all the things we've been learning this week, to give a beautiful relationship between Seiberg-Witten theory and Donaldson theory. Namely, we can take the pre-potential in the $N = 2$ theory and shoot it off to $t \to \infty$ or $t \to 0$. On one end, by applying S duality, one recovers Seiberg-Witten theory. On the other, we recover Donaldson theory.

We've also talked about the idea of *twisting* this week, in the talks of Sam and Yuan. The point is that all this theory is first defined only for $\mathbb{R}^4$ via the superspace construction. Since the supersymmetry operators are spinors, when we try to define the theory on a general 4-manifold, we may not have a section of a spinor bundle to define a supersymmetry operator defined globally on the 4-manifold. The twist allows us to look at SUSY operators which become sections of a different bundle, admitting sections, and in this way we come to study the solutions to Super Yang-Mills (i.e, Donaldson or SW invariants) on a 4-manifold which is not $\mathbb{R}^4$.

2. Superspace for $N = 1$

In Michael's example from yesterday, for $N = 1$ SUSY on dimension 1, fields looked like $\phi(x) + \psi(x)\theta$. We were lucky we had so few θ variables. But today we're going to go hardcore: $N = 1$ to start, but our underlying space will be $\mathbb{R}^4$, so we're going to dimension 4.

2.1. Superfields for $N = 1$ and $V = \mathbb{R}^{1,3}$.

2.1.1. *The basics.* Recall that the underlying manifold of superspace is an affine space modeled on $\mathbb{R}^4$. The super Lie algebra which exponentiates to supserspace is given by

$$V \oplus \Pi(S^*) \cong V \oplus \Pi(S_+ \oplus S_-).$$

Here, $S^* = S_+ \oplus S_-$, and the splitting is such that the pairing matrix

$$\Gamma : S^* \otimes S^* \to V$$

is given by

$$\Gamma = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}.$$

The σ are 2-by-2 complex matrices where, if we choose a basis e_1, e_2 for S_+ and a basis $\bar{\mathrm{e}}_{\dot{1}}, \bar{\mathrm{e}}_{\dot{2}}$ for S_-, there is an isomorphism to $V \cong \mathbb{C}^4 = \mathrm{span}(\{y_\mu\})$ given by

$$\sigma : e_\alpha \otimes \bar{\mathrm{e}}_{\dot{\beta}} \to \sigma^{\mu}_{\alpha\dot{\beta}} y_\mu.$$

These σ^μ are also known as the Dirac matrices.

After exponentiation, we denote the odd coordinates arising from S_+ by $\theta = (\theta^1, \theta^2)$ and from S_- by $\overline{\theta} = (\overline{\theta}^1, \overline{\theta}^2)$. The even coordinates will be written x, so in all the coordinate functions are given by

$$x = (x^0, x^1, x^2, x^3), \qquad \theta^1, \theta^2, \overline{\theta}^1, \overline{\theta}^2.$$

Under the action of $SO(1, 3)$, the θ and $\overline{\theta}$ are spinors. Since a spinor bundle is trivial over $\mathbb{R}^4$, you can think of these as just having two complex coordinates.

Remark 2.1. The importance of this construction–exponentiating the super Lie algebra to a super Lie group–is that we get vector fields on super space via left- and right-invariant vector fields. It is these left-invariant vector fields that act as the supersymmetries $Q_\alpha, \overline{Q}_{\dot{\alpha}}$, and we have (very conveniently!) some right-invariant vector fields $D_\alpha, \overline{D}_{\dot{\alpha}}$ which *do not agree* with the left-invariant vector fields. (This is because our group is not abelian.) The wonderful consequence though, is that the vector fields clearly commute since left- and right-actions of a Lie group commute. We will be using the commutativity of the Q with the D repeatedly.

2.1.2. *The superfields.* What's important really isn't so much the superspace as the superfields.

Superfields are functions on superspace, and for $N = 1$, they take the form

$$\Phi(x, \theta, \overline{\theta}) = \phi(x) + \theta\lambda(x) + \overline{\theta}\,\tilde{\lambda}(x) + \theta^2 m(x) + \overline{\theta}^2\,\tilde{m}(x) +$$

$$\theta\sigma^\mu\,\overline{\theta}\,v_\mu(x) + \theta^2\,\overline{\theta}\,\tilde{\psi}(x) + \overline{\theta}^2\,\theta\psi(x) + \theta^2\,\overline{\theta}^2\,D(x).$$

where you should think of this as a Taylor expansion in the θ and $\overline{\theta}$ variables. Note that:

- Every function of x is a complex-valued function of the even coordinates x.
- $\tilde{\lambda}$ and λ are unrelated—each is a complex-valued function of x, but have no dependence on each other. The twiddle is used only so that I did not have to use another Greek letter to denote the function $\tilde{\lambda}$. Likewise for $\tilde{m}, \tilde{\psi}$.
- The notation $\overline{\theta}$ hasn't shown up before. What I'm doing here is writing the representation $S = \mathbb{R}^{1,3} \otimes \mathbb{C}$ as $\mathbb{C}^2 \otimes \mathbb{C}^2$. The θ^α are a basis for the first $\mathbb{C}^2$, and the $\overline{\theta}^\alpha$ are a basis for the second $\mathbb{C}^2$. ($\alpha \in \{1, 2\}$.)
- there are no terms like $\theta^1\theta^1$—this is because θ^1 is an odd variable, so it squares to zero.
- By θ^2 I mean $2\theta^1\theta^2 = \theta_\alpha\theta^\alpha$, and likewise for $\overline{\theta}^2$.
- by terms like $\theta\psi$ we really mean $\theta^\alpha\psi_\alpha$, and ψ is actually a section of the spinor bundle over $\mathbb{R}^4$.
- σ^μ are just the Dirac matrices. This has appeared in various forms and has also been called Γ. There's a complex, 2 by 2 matrix for every $\mu \in \{0, 1, 2, 3\}$. By convention, the notation $\theta\sigma^\mu\overline{\theta}$ means

  $$\theta\sigma^\mu\overline{\theta} = \theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\,\overline{\theta}^{\dot{\beta}}\,.$$

  It's not necessary to write out the basis of $\theta^\alpha\,\overline{\theta}^{\dot{\beta}}$ in this way, but it will be useful later to verify SUSY.

Remark 2.2. For $N = 2$, the number of θ will increase. In general we let $A \in \{1, \ldots, N\}$ and we define variables $\theta^{\alpha A}$.

Remark 2.3. These have been written out in components, and each component has been written in previous lectures as a pull-back.

I will write the set of all Φ by $\mathcal{F}$. The same symbol $\mathcal{F}$ will show up later in the talk, but as a prepotential. This is an unfortunate abuse of notation, but $\mathcal{F}$ has designated the space of fields in previous talks, so I am following that convention.

I will also define an operation $(\bullet)^\dagger : \mathcal{F} \to \mathcal{F}$, which is defined on functions by complex conjugation, and on the odd variables by

$$\theta_\alpha \mapsto \overline{\theta}_{\dot{\alpha}}, \qquad , \theta_\alpha\theta_\beta \mapsto (\theta_\beta)^\dagger(\theta_\alpha)^\dagger$$

2.2. **Superalgebra action.** Recall that we constructed a super Poincare group yesterday which acts on superspace. I don't want to talk about the group, but the point is that group action generates *right-invariant* and *left-invariant* vector fields. Since the group wasn't abelian, these guys are different vector fields, but since it's left and right actions, the vector fields commute with each other! Let me write down how these vector fields act on a superfield.

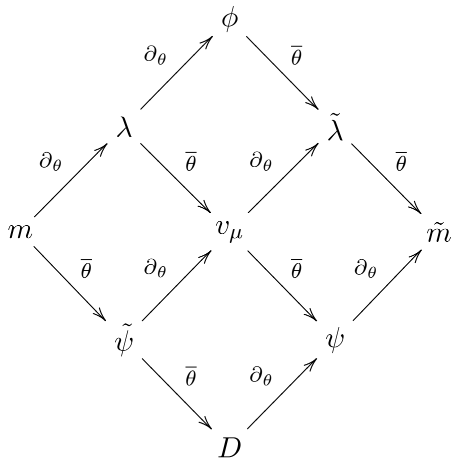
First, the super algebra generators $Q_\alpha, \overline{Q}_{\dot\alpha}$.

$$Q_\alpha = -i(\frac{\partial}{\partial\,\theta^\alpha} - i\sigma^\mu_{\alpha\,\dot\alpha}\,\overline{\theta}^{\dot\alpha}\,\partial_\mu) \qquad \overline{Q}_{\dot\alpha} = i(\frac{\partial}{\partial\,\theta^\alpha} - i\sigma^\mu_{\alpha\,\dot\alpha}\theta^\alpha\,\partial_\mu).$$

These operators have also been written $\tau_{Q_\alpha}, \tau_{\overline{Q}_{\dot\alpha}}$ in previous lectures. For example, if $\Phi = \phi$, then

$$Q_\alpha(\Phi) = -\sigma^\mu_{\alpha\,\dot\alpha}\,\overline{\theta}^{\dot\alpha}\,\partial_\mu\,\phi.$$

The following diagram may be large and confusing, but here are the directions in which the operators $\partial\,/\,\partial\,\theta$ and $\overline{\theta}$ act on components: (I am ignoring coefficients.)



So for instance, the $\tilde m$ term in a superfield Φ is sent to zero by the operator Q_α, while the v_μ term is sent to a sum of a $\tilde\lambda$ component and a ψ component.

3. Supersymmetric actions and chiral fields

3.1. **One verification of SUSY invariance.** So in general you might think it's hard to write down a function on the space of fields which is invariant under a group action. And if the word super appears, it must be uber-difficult. Actually, the super makes it really easy. Here's a SUSY-invariant Lagrangian, pretty much for free!

Proposition 3.1. *The action*

$$\int d^4(x)D(x)$$

is SUSY-invariant.

Here, $D(x)$ is the component of the superfield Φ as written out above. We're assuming that $D(x)$ is compactly supported, not just for integral to be defined, but to use Stokes's theorem in a second. Also, I'm going to do out the computation because I really want to hammer home that these computations are do-able. And the whole philosophy will be to replace the Q_α by D_α, since they are always equal under i^*, as Chris and Michael wrote it.

Proof.

$$\begin{aligned}
Q_\alpha \int d^4x D(x) &= Q_\alpha \int d^4x d^2\theta d^2\,\overline{\theta}\,\Phi(x) \\
&= \int d^4x d^2\theta d^2\,\overline{\theta}\,Q_\alpha \Phi(x) \\
&= \int d^4x d^2\theta d^2\,\overline{\theta} - i(\frac{\partial}{\partial\,\theta^\alpha} - i\sigma^\mu_{\alpha\,\dot\alpha}\,\overline{\theta}^{\dot\alpha}\,\partial_\mu)\Phi(x) \\
&= -\int d^4x d^2\theta d^2\,\overline{\theta}\{\sigma^\mu_{\alpha\,\dot\alpha}\,\overline{\theta}^{\dot\alpha}\,\partial_\mu \Phi(x)\} \\
&= \int d^4x(\text{some coefficients})\,\partial_\mu\,\tilde\psi(x) \\
&= 0.
\end{aligned}$$

Likewise, the $\overline{Q}_{\dot\alpha}$ derivative is an integral over some derivative, and hence equals zero. □

This $D(x)$ is called the **D term** of the superfield Φ.

Now I would like to write down a proposition to motivate the notion of *chirality.*

Proposition 3.2. *Let $\Phi \in \mathcal{F}$ be chiral, let $K(\Phi, \Phi^\dagger)$ be a function in $\Phi, \Phi^\dagger$, and let $W(\Phi)$ be a complex-analytic function in Φ. Then*

$$\int d^4x d^2\theta d^2\,\overline{\theta}\,K(\Phi,\overline{\Phi}) + \int d^4x d^2\theta W(\Phi) + \int d^4x d^2\,\overline{\theta}\,\overline{W}(\overline{\Phi})$$

is SUSY.

Remark 3.3. Some remarks:

- The term containing the K is called the *D term* of the action, and the other term is called the *F term* of the action.
- By a complex analytic function, I mean you write out W as a power series for an analytic function in z for some complex variable z, and to evaluate $W(\Phi)$, we simply replace z by Φ.
- Also, when I say K is a function of $\Phi, \overline{\Phi}$, I mean that K can also have a dependence on $D\Phi$, $\overline{D}\,\overline{\Phi}$, and higher derivatives of Φ.
- Also, an action is SUSY if $Q_\alpha, \overline{Q}_{\dot\alpha}$ annihilate it. This simply means that the action is invariant under the action of the supersymmetry operators.

Before I get to an example, I'd like to discuss chirality a little more.

The right-invariant vector fields $D_\alpha, \overline{D}_{\dot\alpha}$ can be computed to be

$$D_\alpha = \frac{\partial}{\partial\,\theta^\alpha} + i\sigma^\mu_{\alpha\,\dot\alpha}\,\overline{\theta}^{\dot\alpha}\,\partial_\mu, \qquad \overline{D}_{\dot\alpha} = -\frac{\partial}{\partial\,\theta^\alpha} - i\sigma^\mu_{\alpha\,\dot\alpha}\theta^\alpha\,\partial_\mu\,.$$

Definition 3.4. We say that a superfield Φ is *chiral* if $\overline{D}_{\dot\alpha} Q = 0$ for $\dot\alpha = 1, 2$.

This cuts down the number of components in Φ;

Proposition 3.5. *Let Φ be chiral. Then Φ can be written*

$$\Phi(x, \theta, \overline{\theta}) = \phi(y) + \theta^\alpha \psi_\alpha(y) + \theta^2 F(y)$$

under the change of coordinates

$$y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}\,.$$

Remark 3.6. Under this change of coordinates, you need to write down the Taylor expansion of functions like $\phi(y)$ by treating the $i\theta\sigma^\mu\overline{\theta}$ component as an infinitesimal displacement of the x coordinate. So, for instance,

$$\theta^\alpha\psi_\alpha(y) = \theta^\alpha\psi_\alpha(x) + \theta^\alpha(\partial_\mu\psi_\alpha)i\theta\sigma^\mu\overline{\theta}\,.$$

Example 3.7 (Free scalar field). Let $K(\Phi,\Phi^\dagger) = \Phi\Phi^\dagger$. Then you can see that

$$\int d^4x d^2\theta d^2\overline{\theta}\, K = \int d^4x\ \{\phi\Delta\phi + \psi\sigma^\mu\partial_\mu\overline{\psi} + F\overline{F}\}.$$

Note that the first term is the usual scalar field theory, the second term is the usual Dirac equations which introduces spinors into the equations of motion, and the last term in the equation of motion simply sets $F = \overline{F} = 0$. It introduces no new dynamics, and F is simply called an *auxiliary* field for this reason. It is not always the case that F plays no role in the equations of motion, however. Greg will talk about examples in which F will couple to other parts of the fields.

Proof. We already proved the statement for the D term, and one can extend that proof to arbitrary functions using the chain rule.

So let's verify SUSY-invariance for the F term. We have

$$\begin{aligned} Q_\alpha \int d^4x d^2\theta\Phi &= Q_\alpha \int d^4x D_1 D_2(\Phi)|_{\theta=\overline{\theta}=0} \\ &= \int d^4x Q_\alpha D_1 D_2(\Phi)|_{\theta=\overline{\theta}=0} \\ &= \int d^4x D_\alpha D_1 D_2(\Phi)|_{\theta=\overline{\theta}=0} \\ &= \int d^4x 0 \\ &= 0 \end{aligned}$$

since any triple-application of the D_α results in zero. Note that one can replace Q_α by D_α since the two operators agree when restricted to the locus $\theta = \overline{\theta} = 0$. This is in general how most proofs of supersymmetry go—you replace the Q operators by their D counterparts.

The $\overline{Q}$ computation uses the same trick, and now uses chirality as well:

$$\begin{aligned} \overline{Q}_{\dot\alpha} \int d^4x d^2\theta\Phi &= \int d^4x d^2\theta \overline{Q}_{\dot\alpha}\Phi \\ &= \int d^4x d^2\theta \overline{D}_{\dot\alpha}\Phi + \int d^4x d^2\theta(\ldots)\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\partial_\mu\Phi \\ &= \int d^4x d^2\theta 0 + 0 \end{aligned}$$

where we have used the chirality of Φ to set the first integrand to zero, and where the second integral goes to zero because it is the integral of a derivative. □

4. An aside about $N = 2$ SYM

So there's a whole $N = 2$ superspace formalism that's more complicated, and there's a class of superfields that we can talk about, but let me stay at a more distant level. It turns out that an $N = 2$ Lagrangian in SYM will typically look like

$$\frac{1}{2\pi}\,\mathrm{Im}(\int d^4x d^2\theta d^2\overline{\theta}\,\mathrm{Tr}\,\mathcal{F}(\Psi))$$

where $\mathcal{F}$ is some holomorphic function in a vector multiplet Ψ, and it turns out we can write this in$N = 1$ terms using an expansion in θ, to end up with a Lagrangian:

$$\frac{1}{4\pi} ImTr \int d^4x d^2\theta d^2\overline{\theta}\Phi^\dagger e^{2V}\frac{\partial\mathcal{F}(\Phi)}{\partial\,\Phi} \qquad + \qquad \int d^4x d^2\theta\frac{1}{2}\frac{\partial\,\mathcal{F}(\Phi)}{\partial\,\Phi^2}W^\alpha W_\alpha$$

which in general may involve higher derivatives of $\mathcal{F}$, but no matter for now. The point is that this holomorphic function, $\mathcal{F}$, called the *prepotential* determines the whole theory. This will be important in Greg's talk.

5. N=1 Super Yang Mills

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

The Gauge transformations are now given by chiral fields $\Lambda \in \mathcal{F} \otimes \mathfrak{g}$. They transform our fields as follows:

- for V a vector multiplet with $V = V^\dagger$, we have
$$e^{2V} \mapsto e^{i\Lambda^\dagger} e^{2V} e^{-i\Lambda}$$
(this is just a compact way to write it; you'll have to write out the expansions to see how V really transforms.)
- For Φ a chiral multiplet,
$$\Phi \mapsto e^{-i\Lambda}\Phi.$$

When $\mathfrak{g}$ is abelian, you can verify that the even part of the Gauge transformation for a vector multiplet is the familiar Gauge transformation from ordinary Yang-Mills theory.

5.1. **Pure Yang-Mills Action.** Now define

$$W_\alpha = \frac{1}{4}\overline{D}^2(e^{-2V}D_\alpha e^{2V})$$

which is different from the W I wrote in the F term before. (I apologize for the abuse of notation—but this W_α has components.)

Then we set the Yang-Mills action for $N = 1$ to be

$$\int d^4x d^2\theta\,\mathrm{Tr}(W^\alpha W_\alpha).$$

As before, it helps to write out our fields in a more compact form: It turns out that there is a gauge called the *Wess Zumino gauge* in which we can write our real vector multiplet as

$$V = -\theta\,\partial^\mu\,\overline{\theta}\,A_\mu - i\,\overline{\theta}^2\,\theta\lambda + i\theta^2\,\overline{\theta}\,\overline{\lambda} + \frac{1}{2}\theta^2\,\overline{\theta}^2\,D$$

where $\overline{\lambda}$ is really the conjugate of λ. Then one can show that the Lagrangian becomes

$$\int d^4xTr(\frac{-1}{4}F^2 - i\lambda\sigma^\mu\,\partial_\mu\,\overline{\lambda}+\frac{1}{2}D^2).$$

Note that the first term recovers the usual Yang-Mills, the second term is again the introduction of fermions to our equations of motion, and the last D term is auxiliary.

Remark 5.1. Recall from earlier that I claimed any $N = 2$ theory depends only on a holomorphic function $\mathcal{F}$. The pure term I wrote just now is what you obviously get if you set $\mathcal{F} = \Phi^2$. As an exercise, you can try to recover Donaldson theory by studying now the D term contribution (and not just the F term contribution) of the resulting Lagrangian.