# Classical Field Theory



**Aim:** describe the Lagrangian picture of a classical system, & give some basic examples from classical mechanics.

States of a classical system are points in some manifold $\mathcal{M}$, which will come with a symplectic structure $\Omega$. Time evolution is given by a hamiltonian generating a 1-parameter group of diffeomorphisms.

## Lagrangian Systems

Not everything one might want to call a field theory admits a Lagrangian (free chiral scalar field in 2d), but it's a useful picture.

We imagine a large space of fields, and physical states are those fields extremising an action functional (we'll see examples).

A space of fields $\Phi$ will be a space of sections of a fibre bundle. More precisely, have a smooth oriented (for simplicity) $d$-manifold $M$, usually called spacetime (though this is often misleading), & a smooth fibre bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$. Then $\Phi$ is the space of smooth sections $M \xrightarrow{\phi} E$.

## Examples

- $E = \begin{matrix} M \times X \\ \downarrow \\ M \end{matrix}$ trivial bundle. Then $\Phi = \mathrm{Map}(M, X)$. These will be the fields in a $\sigma$-model.

  If $X = \mathbb{R}$ or $\mathbb{C}$, these are called scalar fields.
- $\begin{matrix} S \\ \downarrow \\ M \end{matrix}$ a spinor bundle, associated to the frame bundle by a spin (or spin$^c$) rep: spinor fields
- $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$ a principal $G$-bundle, $\begin{matrix} \mathrm{ad}\, P \\ \downarrow \\ M \end{matrix} = P \times_G \mathfrak{g}$. One has the affine space bundle

  $\begin{matrix} T^*M \otimes \mathrm{ad}\, P \\ \downarrow \\ M \end{matrix}$. Sections are connections: pure gauge fields

  (one probably wants to take all $P$ at once...)
- metrics

Given $\Phi$, a Lagrangian density $\mathcal{L}(\phi)$ is a smooth function $\Phi \longrightarrow \Omega^d(M)$ that is local, i.e. there is some $k$ so that the value of $\mathcal{L}(\phi)$ at $m \in M$ depends only on the $k$-jet of $\phi$.
i.e. $\mathcal{L}$ factors through a map of bundles $J^k(E) \longrightarrow \Omega^d(M)$ over $M$.

The action functional is (using orientable).

$$S(\phi) = \int_M \mathcal{L}(\phi).$$

(usually divergent, but one can compute if on compact sets)

Euler Lagrange

The space of classical solutions $\mathcal{M} \subseteq \Phi$ is given by those states $\phi_0$ extremising the action. Roughly, for $\delta\phi$ a small perturbation so that $\delta\mathcal{L} = \mathcal{L}(\phi_0 + \delta\phi) - \mathcal{L}(\phi_0)$ is compactly supported, say $\phi_0$ is extremal if $\delta S(\phi_0) = \int \delta\mathcal{L}(\phi_0) = 0$ for any such $\delta\phi$. $\delta\mathcal{L}$ is called the first variation.

We'll see an exact Lagrangian structure on $\mathcal{M}$ in some examples.

The Classical particle (1d $\sigma$-model)

Consider a particle moving freely through $X = \mathbb{R}^n$. We know (Newton's laws), that the trajectories of the particle are straight lines $x(t)$ with $\ddot{x}(t) = 0$. One can produce this via a Lagrangian formulation.

The fields here are possible trajectories $x : \mathbb{R} \longrightarrow X$ (i.e. $E = \mathbb{R} \times X$ trivial bundle).

We try to extremise kinetic energy $T(x) = \frac{1}{2}mv^2 = \frac{1}{2}m|\dot{x}|^2$

Here we use a Riemannian metric on $X$.

Lagrangian density is $\frac{1}{2}m|\dot{x}|^2\, dt$.

So $S(x) = \int \frac{1}{2} m |\dot{x}|^2 dt$. We'll compute the first variation.

Look at $\delta x$ with compact support in $(t_0, t_1)$.

$$\delta \mathcal{L}(x) = m < \dot{x}, \delta \dot{x} > dt$$
$$= -m < \ddot{x}, \delta x > dt + d(m < \dot{x}, \delta x >) \quad \text{by product rule}$$

$$\Rightarrow \delta S(x) = -m \int_{t_0}^{t_1} < \ddot{x}, \delta x > dt + m\left( < \dot{x}(t_1), \delta x(t_1) > - < \dot{x}(t_0), \delta x(t_0) > \right)$$

So $\delta S$ is extremised if $\ddot{x} = 0$.

Write $\gamma(t) = m(\dot{x}(t) \cdot dx(t))$. The above implies it is independent of $t$ as a 1-form on $\mathcal{M}$.

Put
$$\Omega = d\gamma \in \Omega^2(\mathcal{M})$$
$$= m(d\dot{x} \wedge dx)$$

In fact, $\gamma$ is a connection on the trivial $\mathbb{R}$-torsor over $\mathcal{M}$, with curvature $\Omega$.

Note:

Choose $t_0$. Then
$$\mathcal{M} \longrightarrow TX \quad \text{is an isomorphism.}$$
$$x \longmapsto (x(t_0), \dot{x}(t_0))$$

The standard symplectic structure on $TX$ pulls back to give $\Omega$.

One can extend this picture to include potential: If $V$ is a functional on $X$, consider $\mathcal{L}(x) = (T(x) - V(x))dt$

Then one recovers Newton's $2^{nd}$ law: $\nabla V(x) = m\ddot{x}$.

## Relativistic Particle

Now, let $X = \mathbb{R}^{1,n-1}$, Minkowski space with its standard metric: $c^2(dt)^2 - (dx_1)^2 - \dots - (dx_{n-1})^2$

Note we've picked a basis: Inertial frame.

So fields now look like
$$x : \mathbb{R} \longrightarrow X$$
$$\tau \longmapsto (t(\tau), x_1(\tau), \dots, x_{n-1}(\tau))$$

Recall: in special relativity, the energy of a particle is given by $\gamma mc^2$, where $\gamma = (1 - \frac{v^2}{c^2})^{-1/2}$. One chooses a related Lagrangian, in order to recover this as the Hamiltonian.

$$\mathcal{L}(x) = -mc\left( \frac{dx}{d\tau}, \frac{dx}{d\tau} \right)^{1/2} d\tau.$$

Note if $t(\tau) = \tau$, this becomes
$$-mc^2 \sqrt{1 - \frac{v^2}{c^2}}.$$

For physical particles, $\langle \frac{dx}{d\tau}, \frac{dx}{d\tau} \rangle > 0$ : timelike.

One again solves the E-L equations, & finds particles move along straight lines. Note if $v << c$,

$\sqrt{1 - \frac{v^2}{c^2}} \approx 1 - \frac{1}{2}\frac{v^2}{c^2}$, so the Lagrangian is approximately

$(-mc^2 + \frac{1}{2}mv^2)\, dt$, which recovers the classical particle.

Boundary term is $\gamma = -mc \dfrac{(\dot{x}(\tau) \cdot dx(\tau))}{|\dot{x}(\tau)|}$, again independent of $\tau$

One can produce a potential, in the form of an electromagnetic field $\alpha$ on $X$ : a connection on an $\mathbb{R}$-torsor. One considers Lagrangian density $-mc \langle \frac{dx}{d\tau}, \frac{dx}{d\tau} \rangle^{1/2} d\ell - q x^*(\alpha)$.

## Symmetries

We produced $\mathcal{M}$ with a canonical $\mathbb{R}$-torsor with connection, but we could've done so in different ways. e.g there were symmetries of the target space $X$ that preserved this data.

e.g :

Non-relativistic particle. The group $\mathrm{Isom}(X)$ acted on $X$. Lets look at generators of the Lie algebra: vector fields on $X$, hence on the canonical torsor $P \to \mathcal{M}$.

Can associate to such a $\xi$, a charge $Q \in C^\infty(\mathcal{M})$, such that $Q = i(\xi)\gamma$. (In fact, there is a correspondence between them).

Here, $\gamma = m \langle \dot{x} \cdot dx \rangle$. $\quad \xi = \frac{\partial}{\partial x^i} \Rightarrow Q = p_i = m\,\delta_{ij}\,\dot{x}^j$ momentum

$\xi = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i} \Rightarrow Q = M^i_j = m\,\delta_{jk}(x^i \dot{x}^k - x^k \dot{x}^i)$

corresp to rotation

angular momentum

One can do a similar thing for a relativistic particle. (5)

One recovers energy: the Hamiltonian from — the time translation symmetry.
$-\frac{d}{dt}$.

These charges $Q$ are conserved in time: they live on $\mathcal{M}$, not $P$.

Very roughly: Noether's theorem says: to any infinitesimal symmetry of $\mathcal{L}$, there is a conserved quantity $Q$. Usually it'll be a current, i.e. a distribution on $\Omega^1(M)$.